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The role of spatial analyticity in the local alignment of vorticity directions in 3D viscous fluids

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Abstract. We study local spatial dynamics of the vorticity directions in the 3D Navier–Stokes model. More precisely, utilizing the spatial analyticity of solutions, we obtain estimates of the 2D Hausdorff measure of the level sets arising in the local alignment.

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1. Introduction

In [C2], Constantin derived an integral representation of the stretching factor in the evolution of vorticity magnitude that clearly demonstrates how the local alignment of vorticity directions, both parallel and antiparallel, depletes the nonlinearity in 3D Navier–Stokes equations (NSE). It is important that the antiparallel alignment also depletes the nonlinearity because antiparallel vortex line pairing has been observed both in numerical simulations and physical experiments. In [CoFe], under a precise assumption on the local alignment, a regularity result for 3D NSE was rigorously established. The known regularity results for the direction of vorticity can be found in [C1, C2, CoFe]—they are essentially local L^2 bounds on the gradient of the vorticity direction in the regions of high vorticity magnitude. Analogous L^∞ bounds would imply the assumption needed for the proof of global regularity presented in [CoFe].

The above results are a strong motivation for studying the local spatial dynamics of the vorticity directions. In this paper, we consider a regular solution on some time interval and show how the viscosity through the spatial analyticity of solutions (regular solutions are automatically spatially analytic, see e.g. [FT, GK1, M]) influences the dynamics of the local alignment. The main result is measure-theoretic in nature and is presented in theorem 4.1. The result is local and is valid on the balls in the vortical region with radii less than or comparable to the radius of spatial analyticity. Thus, the radius of spatial analyticity emerges as a significant parameter in the study of the local spatial dynamics of the vorticity directions in 3D viscous fluids.

2. Notation

We consider the NSE in $\Omega \subseteq \mathbb{R}^3$ with potential forces where Ω can be either $\mathbb{R}^3$ or $[0, L]^3$ with periodic boundary conditions or a bounded smooth domain in $\mathbb{R}^3$ with Dirichlet boundary

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conditions.

$$\frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla\pi = 0, \qquad (2.1)$$
$$\nabla\cdot u = 0$$

for $(x, t) \in \Omega \times (0, \infty)$, supplemented with the initial condition

$$u(x, 0) = u_0(x), \qquad x \in \Omega, \qquad (2.2)$$

where u is the velocity, π is the pressure, $\nu > 0$ is the viscosity and $\nabla\cdot u_0 = 0$.

For the properties of solutions in the whole space the reader is referred to [FJR, K], and for the bounded domains to [CF].

The vorticity $\omega = \nabla\times u$ satisfies the equation

$$\frac{\partial\omega}{\partial t} - \nu\Delta\omega + u\cdot\nabla\omega = \omega\cdot\nabla u. \qquad (2.3)$$

Throughout this paper, c, c_i will denote positive universal constants that may differ from line to line.

3. Spatial analyticity

Henceforth, we assume that $\Omega = [0, L]^3$—the assumption implies no loss of generality since similar analyticity results are or could be obtained for the other types of domains as well [CRT, GK1, GK2].

Let

$$R_0 = \frac{L}{\nu}\|u_0\|_{L^\infty}$$

denote a nondimensional quantity that could be thought of as the Reynolds number of the initial configuration. Then we have the following analyticity result for u.

Theorem 3.1 ([GK1]). *Let $T = \frac{L^2}{c\nu R_0^2(1+\log_+ R_0)^2}$. Then, the solution u of (2.1), (2.2) satisfies the following property: for every $t \in (0, T)$, u is a restriction of an analytic function $u(x, y, t) + \mathrm{i}v(x, y, t)$ in the region*

$$\mathcal{D}_t = \{x + \mathrm{i}y \in \mathbb{C}^3 : |y| \leqslant c^{-1}(\nu t)^{1/2}\}.$$

Moreover,

$$\|u(\cdot, y, t)\|_{L^\infty} + \|v(\cdot, y, t)\|_{L^\infty} \leqslant c\frac{\nu R_0}{L},$$

for $t \in (0, T)$ and $(x, y) \in \mathcal{D}_t$.

Now, consider a time interval $\Delta T = [T_1, T_2]$, where

$$0 < T_1 = \frac{L^2}{c_1\nu R_0^2(1+\log_+ R_0)^2} < \frac{L^2}{c_2\nu R_0^2(1+\log_+ R_0)^2} = T_2 < T.$$

Then, we have a uniform lower bound on the analyticity radius on ΔT:

$$\tau_{\Delta T} = \frac{L}{cR_0(1+\log_+ R_0)}. \qquad (3.1)$$

Before we proceed, we need another piece of notation. Let $S_b = \{x + \mathrm{i}y \in \mathbb{C}^3 : |y| \leqslant b\}$, for some $b > 0$. Then, for a complex-valued function F analytic on S_b, set

$$\|F\|_{H^\infty(S_b)} = \sup_{z\in S_b}|F(z)|.$$

We are now ready to derive the needed analyticity properties of ω on ΔT.

Recall that

$$\omega = \nabla \times u = (\partial_2 u_3 - \partial_3 u_2, -\partial_1 u_3 + \partial_3 u_1, \partial_1 u_2 - \partial_2 u_1). \tag{3.2}$$

We can complexify the relation (3.2) and for the complexified vorticity $\omega = (\omega_1, \omega_2, \omega_3)$. The Cauchy integral representation formula for derivatives combined with (3.1) and theorem 3.1. yield the following result.

Corollary 3.2. *Let $\tau = \frac{\tau_{\Delta T}}{2}$. Then,*

$$\max_{t\in\Delta T} \|\omega_j(t)\|_{H^\infty(S_\tau)} \leqslant \frac{c}{\tau} \max_{t\in\Delta T} \max_{|y|\leqslant 2\tau} (\|u(\cdot, y, t)\|_{L^\infty} + \|v(\cdot, y, t)\|_{L^\infty})$$
$$\leqslant c R_0^2 (1 + \log_+ R_0) \frac{\nu}{L^2} = M(R_0, \nu, L) = M,$$

for $j = 1, 2, 3$.

Remark 3.3. *We could have also derived the desired analyticity properties of ω directly from the vorticity equation (2.3) using the method presented in [GK1].*

4. Spatial analyticity and the local alignment of vorticity directions

We first briefly explain how the local alignment of vorticity directions depletes the nonlinearity in 3D NSE, closely following the presentation of [C2, CoFe].

Consider the vortical region

$$\Omega^* = \{x \in \Omega : |\omega(x)| > 0\}.$$

Then, the direction of vorticity

$$\xi(x) = \frac{\omega(x)}{|\omega(x)|}$$

is well defined and we can introduce

$$\alpha(x) = S(x)\xi(x) \cdot \xi(x),$$

where

$$S(x) = \tfrac{1}{2}(\nabla u(x) + (\nabla u(x))^T)$$

is the strain matrix.

A remarkable integral representation formula for α was derived in [C2] (here $\Omega = \mathbb{R}^3$):

$$\alpha(x) = \frac{3}{4\pi} P.V. \int_{\mathbb{R}^3} D(\hat{y}, \xi(x+y), \xi(x)) |\omega(x+y)| \frac{\mathrm{d}y}{|y|^3}, \tag{4.1}$$

where $\hat{y} = \frac{y}{|y|}$ and the geometric factor D is given by

$$D(e_1, e_2, e_3) = (e_1 \cdot e_2)(\mathrm{Det}(e_1, e_2, e_3)), \tag{4.2}$$

for any three unit vectors e_1, e_2, e_3. If $\xi(x)$ and $\xi(x+y)$ are either parallel or antiparallel then $D = 0$, and since it can be shown that the contribution of the region $\{|y| > d\}$, $d > 0$, to the integral in (4.1) is *a priori* bounded in terms of the initial velocity and d the *local* alignment of vorticity directions controls α. The dynamical significance of α is that it represents the stretching factor in the evolution of the vorticity magnitude, and thus the local alignment of vorticity directions plays the crucial role in a possible blow-up.

Now, we go back to our regular, spatially analytic ω on the time interval ΔT. Since we are only interested in the local interactions in the regions of high vorticity magnitude, we will consider a ball

$$B(x_0, r) = \{x \in \mathbb{R}^3 : |x - x_0| < r\} \subseteq \Omega^1 = \{x \in \Omega : |\omega(x)| \geqslant 1\},$$

such that $r \leqslant \frac{\tau}{5}$. Also, we assume that $M = M(R_0, \nu, L) \geqslant 1$. Recall that τ is a uniform lower bound on the radius of spatial analyticity of ω on ΔT.

For any two nonzero vectors $x, y \in \mathbb{R}^3$, denote by $\phi(x, y)$ the angle between x and y. Then

$$\cos\phi(\omega(x), \omega(x_0)) = \frac{\omega(x)\cdot\omega(x_0)}{|\omega(x)||\omega(x_0)|} \tag{4.3}$$

is well defined for $x \in B(x_0, r)$.

For fixed $\eta \in [0, 1)$ consider

$$F_\eta(x) = \frac{(\omega(x)\cdot\omega(x_0))^2}{(\omega(x)\cdot\omega(x))(\omega(x_0)\cdot\omega(x_0))} - \eta, \tag{4.4}$$

for $x \in B(x_0, r)$.

We will study the zero sets of the family F_η, $\eta \in [0, 1)$. Notice that η close to 1 corresponds to the local alignment, both parallel and antiparallel, of the vorticity directions near x_0.

Theorem 4.1. *Let $\eta \in [0, 1)$, and $B(x_0, r) \subseteq \Omega^1 = \{x \in \Omega : |\omega(x)| \geqslant 1\}$, $r \leqslant \frac{\tau}{5}$, where τ denotes the uniform lower bound on the radius of spatial analyticity of ω. Then,*

$$\mathcal{H}^2\{x \in B(x_0, r) : \cos^2\phi(\omega(x), \omega(x_0)) = \eta\} \leqslant F\left(M^2\frac{1}{1-\eta}\right) r^2,$$

where F is a universal, *monotone function, $F(z) \to \infty$, $z \to \infty$, $\mathcal{H}^2$ denotes 2D Hausdorff measure and $M = M(R_0, \nu, L)$ from corollary 3.2.*

In the proof, we shall use the following simple consequence of Jensen's formula that was also utilized in [G] to estimate the number of rapid spatial oscillations of solutions of the Kuramoto–Sivashinsky equation.

Lemma 4.2. *Let $z_0 \in \mathbb{C}$, $r > 0$, and let g be analytic on $\{z \in \mathbb{C} : |z - z_0| \leqslant 2r\}$ such that $g(z_0) \neq 0$. Then,*

$$\mathrm{card}\{z \in \mathbb{C} : |z - z_0| \leqslant r, g(z) = 0\} \leqslant c \log \frac{\max_{|z-z_0|\leqslant 2r} |g(z)|}{|g(z_0)|}.$$

Proof of theorem 4.1. All the estimates will be uniform in $t \in \Delta T$; consequently we will omit the t variable in the computations.

Since we are in Ω^*, $F_\eta(x) = 0$ is equivalent to

$$G_\eta(x) = (\omega(x)\cdot\omega(x_0))^2 - \eta(\omega(x)\cdot\omega(x))(\omega(x_0)\cdot\omega(x_0)) = 0, \tag{4.5}$$

for $x \in B(x_0, r)$.

By corollary 3.2 G_η can be extended to an analytic function on S_τ. Denote the complexified vorticity by $\omega = (\omega_1, \omega_2, \omega_3)$ and set $\omega(x_0) = (\beta_1, \beta_2, \beta_3)$. Then,

$$|G_\eta(z)| = |(\omega_1\beta_1 + \omega_2\beta_2 + \omega_3\beta_3)^2 - \eta(\omega_1^2 + \omega_2^2 + \omega_3^2)(\beta_1^2 + \beta_2^2 + \beta_3^2)| \leqslant cM^2|\omega(x_0)|^2, \tag{4.6}$$

for $z \in S_\tau$, where $M = M(R_0, \nu, L)$ is the same as in corollary 3.2.

Now, consider

$$H^0_{\eta,a}(z) = G_\eta(x_0 + za),$$

where a is a unit vector in S^2, and $z \in \{w \in \mathbb{C} : |w| \leqslant 2r\} = D_{2r}$. By the choice of r, $H^0_{\eta,a}$ is analytic on D_{2r} and applying lemma 4.2

$$\operatorname{card}\{x \in (-r, r) : H^0_{\eta,a}(x) = 0\} \leqslant c \log \frac{\max_{|z|\leqslant 2r} |H^0_{\eta,a}(z)|}{|H^0_{\eta,a}(0)|}. \tag{4.7}$$

Since

$$H^0_{\eta,a}(0) = G_\eta(x_0) = |\omega(x_0)|^4 - \eta|\omega(x_0)|^4 = |\omega(x_0)|^4(1-\eta)$$

and also

$$\max_{|z|\leqslant 2r} |H^0_{\eta,a}(z)| \leqslant \max_{z\in S_\tau} |G_\eta(z)| \leqslant cM^2|\omega(x_0)|^2,$$

(4.7) implies that

$$\begin{aligned}\operatorname{card}\{x \in (x_0 - ra, x_0 + ra) : G_\eta(x) = 0\} &\leqslant c_1 \log\left(c_2 \frac{M^2}{|\omega(x_0)|^2}\frac{1}{1-\eta}\right)\\ &\leqslant c_1 \log\left(c_2 M^2 \frac{1}{1-\eta}\right),\end{aligned} \tag{4.8}$$

uniformly in $a \in S^2$.

Since G_η is analytic, by the structure theorem, the set $\{x \in B(x_0, r) : G_\eta(x) = 0\}$ is a union of the singular set S^*, $\mathcal{H}^2(S^*) = 0$, and countably many 2D analytic manifolds M_k, $k = 1, \dots, \infty$ (see [Ku1, Ku2], where this method was employed, and references therein). In order to estimate the surface measure $\mathcal{H}^2$ via integration over S^2 of 1D estimates of type (4.8), we need uniform estimates on the rays emanating from four *noncoplanar* points. The reason is that the integration will give a bound of type

$$\mathcal{H}^2(\cup_{k=1}^\infty M_k) \leqslant A(\psi^*) \int_{S^2} \operatorname{card}\{x \in B(x_0, r) \cap [x_o, x_o + ra) : G_\eta(x) = 0\} \,\mathrm{d}a\, r^2,$$

where ψ^* is the minimal angle of the normals of the manifolds M_k with the planes orthogonal to the rays $x_0 + a$, $a \in S^2$, and A is a *universal*, monotone function, $A(\psi*) \to \infty$, $\psi^* \to 0$. Nonzero ψ^* may not exist, and hence the above estimate may explode; however, if we have four noncoplanar points, we can split $\cup_{k=1}^\infty M_k$ in four parts such that the appropriate angles will be greater or equal to a certain ψ^*.

We will now construct the other three points, derive appropriate 1D estimates, and *explicitly* estimate the minimal angle ψ^* in terms of the parameters of the problem.

Let $\epsilon = \frac{1-\eta}{2}$. Then, for $x \in B(x_0, r)$,

$$1 - \cos^2\phi(\omega(x), \omega(x_0)) \leqslant \|\nabla(\cos^2\phi(\omega(\cdot), \omega(x_0)))\|_{L^\infty(B(x_0,r))}|x - x_0|. \tag{4.9}$$

A simple computation (taking into account that we are in Ω^1) shows that

$$\|\nabla(\cos^2\phi(\omega(\cdot), \omega(x_0)))\|_{L^\infty(B(x_0,r))} \leqslant c\|\nabla\omega\|_{L^\infty(B(x_0,r))}, \tag{4.10}$$

and $\|\nabla\omega\|_{L^\infty(B(x_0,r))}$ can be estimated from theorem 3.1 using the Cauchy formula.

Hence,

$$1 - \cos^2\phi(\omega(x), \omega(x_0)) \leqslant c\frac{M}{\tau}|x - x_0| \leqslant \epsilon, \tag{4.11}$$

as long as $|x - x_0| \leqslant \min\{c\frac{\tau}{M}\epsilon, r\} = \rho(\epsilon, r)$.

We choose x_1, x_2 and x_3 such that $|x_0 - x_i| = \rho(\epsilon, r)$, $i = 1, 2, 3$, and $\{x_0, x_1, x_2, x_3\}$ is an orthogonal frame. This is the optimal choice in $B(x_0, \rho(\epsilon, r))$ because the points are as noncoplanar as possible.

A simple geometric argument shows that we can now divide $\cup_{k=1}^{\infty} M_k$ into four sets S_0, S_1, S_2, S_3 such that the minimal angle ψ^* between the planes orthogonal to $[x_i, x_i + a)$, $a \in S^2$ and the normals of the manifolds in S_i, $i = 0, 1, 2, 3$, satisfies

$$\tan \psi^* = \frac{c_1 \rho(\epsilon, r)}{r + c_2 \rho(\epsilon, r)} \geqslant c \frac{1}{M} \epsilon = c \frac{1}{M}(1 - \eta) \geqslant c \frac{1}{M^2}(1 - \eta). \tag{4.12}$$

Consider now the functions

$$H^i_{\eta,a}(z) = G_\eta(x_i + za),$$

for $i = 1, 2, 3$, where $a \in S^2$, and $z \in \{w \in \mathbb{C} : |w| \leqslant 4r\}$.

Computations analogous to the derivation of (4.8) imply

$$\begin{aligned} \operatorname{card}&\{x \in (x_i - 2ra, x_i + 2ra) : G_\eta(x) = 0\} \\ &\leqslant c_1 \log \left(c_2 \frac{M^2}{|\omega(x_i)|^2} \frac{1}{\cos^2 \phi(\omega(x_0), \omega(x_i)) - \eta} \right) \\ &\leqslant c_1 \log \left(c_2 M^2 \frac{1}{\cos^2 \phi(\omega(x_0), \omega(x_i)) - \eta} \right), \end{aligned} \tag{4.13}$$

$i = 1, 2, 3$, uniformly in $a \in S^2$.

Utilizing (4.11),

$$\begin{aligned} \operatorname{card}\{x \in (x_i - 2ra, x_i + 2ra) : G_\eta(x) = 0\} &\leqslant c_1 \log \left(c_2 M^2 \frac{1}{1 - \eta - \epsilon} \right) \\ &\leqslant c_1 \log \left(c_2 M^2 \frac{1}{1 - \eta} \right), \end{aligned} \tag{4.14}$$

$i = 1, 2, 3$, uniformly in $a \in S^2$.

We are now ready to estimate the surface measure.

$$\begin{aligned} \mathcal{H}^2\{x \in B(x_0, r) : G_\eta(x) = 0\} &\leqslant A(\psi^*) \sum_{i=0}^{3} \int_{S^2} \operatorname{card}\{x \in (x_i - (1 + \min\{1, i\}) ra, x_i \\ &\quad + (1 + \min\{1, i\}) ra) \cap S_i\} \, \mathrm{d}a \, r^2 \\ &\leqslant c_1 \, A\left(\tan^{-1} \left(c \frac{1}{M^2}(1 - \eta) \right) \right) \log \left(c_2 M^2 \frac{1}{1 - \eta} \right) r^2 \\ &= F\left(M^2 \frac{1}{1 - \eta} \right) r^2, \end{aligned} \tag{4.15}$$

concluding the proof of theorem 4.1. □

For simplicity of the exposition, we now assume that L and ν are fixed and treat them as constants, so that $R_0 = c\|u_o\|_{L^\infty}$.

Then the estimate (4.15) becomes

$$\mathcal{H}^2\{x \in B(x_0, r) : \cos^2 \phi(\omega(x), \omega(x_0)) = \eta\} \leqslant F\left(c R_0^4 (1 + \log_+ R_0)^2 \frac{1}{1 - \eta} \right) r^2. \tag{4.16}$$

Hence,

$$\mathcal{H}^2\{x \in B(x_0, r) : \cos^2 \phi(\omega(x), \omega(x_0)) = \eta\} \leqslant G(R_0) r^2, \tag{4.17}$$

for a *universal*, monotone function G, $G(z) \to \infty$, $z \to \infty$, as long as

$$\frac{1}{1 - \eta} \leqslant \mathrm{e}^{R_0},$$

which is equivalent to

$$|\sin\phi(\omega(x),\omega(x_0))| \geqslant \mathrm{e}^{-\frac{1}{2}R_0}. \tag{4.18}$$

Define

$$C(x_0,r) = \{x \in B(x_0,r) : |\sin\phi(\omega(x),\omega(x_0))| < \mathrm{e}^{-\frac{1}{2}R_0}\}.$$

Then, the above considerations can be summarized in the following corollary.

Corollary 4.3. *Assume that* $B(x_0,r) \subseteq \Omega^1 = \{x \in \Omega : |\omega(x)| \geqslant 1\}$, *and*

$$r \leqslant \frac{c_1}{R_0(1+\log_+ R_0)} = \frac{\tau}{5}.$$

Then,

$$\mathcal{H}^2\{x \in B(x_0,r) - C(x_0,r) : \cos^2\phi(\omega(x),\omega(x_0)) = \eta\} \leqslant G(R_0)\, r^2, \tag{4.19}$$

for a universal, *monotone function* G, $G(z) \to \infty$, $z \to \infty$; *i.e. we have a uniform estimate on the spatial complexity of the vorticity directions in the complement of the exponentially (in* R_0*) aligned region* $C(x_0,r)$.

Remark 4.4. *From [C2, CoFe] we know that the aligned regions are 'good', i.e. the alignment depletes the nonlinearity. Corollary 4.3 states that, in the complement of an exponentially (in* R_0*) aligned region, at least we have uniform control on the spatial complexity of the vorticity directions.*

Remark 4.5. *If the viscosity is not present, i.e. in the case of the 3D Euler equations, to obtain spatial analyticity we have to assume the analytic initial data and even then the radius of spatial analyticity decays exponentially in time [LO]; consequently the analogous result in the absence of viscosity would be much weaker.*

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