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Gevrey regularity for a class of water-wave models

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ABSTRACT

Local well posedness for a class of higher-order nonlinear dispersive partial differential equations is obtained in spaces of functions analytic on a strip around the real axis. The proof relies on estimates in space–time norms adapted to the linear part of the equation. The class of equations in view contains a number of equations arising in the modeling of waves in fluids and also in other applications.

1. Introduction

Consideration is given to the local well posedness of the Cauchy problem associated with a class of nonlinear dispersive wave equations. The general class of equations in view includes various fifth- and higher-order equations appearing in a number of situations as models for surface and internal water waves, as well as in other modeling situations. The focus will be on solutions that are real-analytic in the spatial variable. While parabolic equations such as the heat or the Navier–Stokes equations display analytic smoothing of non-analytic data, dispersive equations do not usually feature analytic solutions unless the initial data are analytic. But even then, the analyticity of the solutions for positive times is a delicate question. The study of this subject was initiated by Kato and Masuda [1] who focused on a fairly general class of equations of the form

$$u_t + f(u)_x + Lu_x = 0, \tag{1.1}$$

where L is an unbounded self-adjoint operator satisfying some modest requirements. They showed that if the initial data are analytic in some strip about the real axis, then the solution continues to be analytic in the space variable, even though the width of the strip may shrink. A close study of the argument in [1] reveals that the width of the strip (also called the radius of analyticity) may at worst decrease doubly exponentially. This lower bound was recently improved by Bona and Grujić in [2] to a simple exponential lower bound. More recently, it has in fact been shown that in the special case of the generalized Korteweg–de Vries (KdV) equation

$$u_t + u^p u_x + u_{xxx} = 0, \tag{1.2}$$

the radius of analyticity will decrease only algebraically [3]. The results mentioned so far are global in time, and as of now, there do not exist any sharp results, or explicit upper bounds for the time development of the radius of analyticity. There

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have been concurrent campaigns by several groups [4–7] investigating the local-in-time well posedness of the generalized KdV equation in classes of analytic functions. In fact, it was recently proved by the authors that the generalized KdV equation can be solved on an invariant domain [8]. Accordingly, if initial data u_0 are given which are analytic and square-integrable in some strip about the real axis, then the solution $u(\cdot, t)$ will continue to be analytic in the same strip at least locally in time.

At this point, it is natural to ask what happens for other equations, as the KdV equation is but one of a wide class of models of the form (1.1). A very general study of well posedness for higher-order equations in weighted Sobolev spaces was conducted by Kenig, Ponce and Vega in [9]. Various other authors have also studied some particular cases of fifth-order nonlinear dispersive equations [10–13]. In spaces of analytic functions, there seems to be only the work of Hayashi [14], in which he considered spaces with a weight in the spatial variable, in this way limiting the application to initial data that are rapidly decaying. Our aim is now to present a sharpened theory for local well posedness for a class of equations in spaces of analytic functions. In particular, our theory will be applicable to initial data without rapid spatial decay. The equation under study is of the general form

$$\partial_t u + \sum_{k=0}^{N} \sum_{l=0}^{N} \partial_x^k \left\{ \sum_{m=0}^{N-k} (\partial_x^m u)\, P_{k,l,m}(\partial_x^l u) \right\} + \sum_{k=1}^{N} b_k \partial_x^{2k+1} u = 0, \tag{1.3}$$

with initial data $u(x, 0) = u_0$ in a class of functions analytic in a symmetric strip around the real axis. The functions $P_{k,l,m}(u)$ are polynomials in u without constant term and b_k are real constants. The highest-order term in the equation is the dispersive term $\partial_x^\alpha u$, where $\alpha = 2N + 1$. It is clear that Eq. (1.3) includes as special cases the generalized KdV equation (1.2). As it will turn out, the Eq. (1.3) also contains many fifth and higher-order model equations that have appeared in different physical situations over the years. Some of these equations will be recorded in Section 6. It should be mentioned here that understanding the analyticity properties of solutions of these equations is not simply an end in itself. As explained in [15,16], analyticity of solutions can be important in the numerical study of equations like (1.2). For instance, in the case of spectral methods, the spatial analyticity of the solution has a profound impact on the convergence rate of spectral projections. Analyticity – when available – may also be exploited to gain improved insight into the physical systems underlying these and other model equations. This was exhibited in [17] in the case of the Navier–Stokes equations.

Before closing the introduction, it is in order to state the main result of this article. For $\sigma > 0$ and $s \in \mathbb{R}$, define $G^{\sigma,s}$ to be the subspace of $L^2(\mathbb{R})$ for which the quantity

$$\|u_0\|^2_{G^{\sigma,s}} = \int (1 + |\xi|)^{2s} e^{2\sigma(1+|\xi|)} |\hat{u}_0(\xi)|^2 \, d\xi$$

is finite.

Theorem 1. *Let $s \geq 2N + \frac{1}{2}$. For initial data in $G^{\sigma,s}$, there exists a positive time T, such that the initial-value problem associated to (1.3) is well posed in the space $C([-T, T], G^{\sigma,s})$.*

Well posedness is understood in the usual Hadamard sense, including existence, uniqueness and continuous dependence on the initial data. $C([-T, T], G^{\sigma,s})$ denotes the space of continuous functions from the time interval $[-T, T]$ into $G^{\sigma,s}$. An argument of Paley–Wiener type may be employed to see that functions in $G^{\sigma,s}$ are restrictions to the real axis of functions analytic in a strip of width 2σ about the real axis [18]. Thus Theorem 1 is really a statement about existence of solutions that are analytic in the spatial variable. The theorem will be proved by exploiting the dispersive smoothing inherent in the linear part of Eq. (1.3), as observed in [19–21]. As will be seen, a combination of the $G^{\sigma,s}$-norm and a Bourgain type norm in space–time Fourier transform variables will allow us to carry through the proof of Theorem 1.

The paper is organized as follows. In Section 2, we define some function spaces to be used in the proof of Theorem 1. Section 3 contains the proof of the crucial multilinear estimates, and in Sections 4 and 5, Theorem 1 is proved via a contraction argument. Finally Section 6 contains some examples, as well as an extension of Theorem 1 to non-local equations.

2. Functional setting and linear estimates

Throughout this paper, standard notation is used. Thus the Lebesgue classes on the real line are denoted by L^p, while the following notation is used to denote the L^p–L^q space–time norms:

$$\|v\|_{L_p L_q} = \left\{ \int \left| \int |v(x, t)|^q \, dt \right|^{\frac{p}{q}} dx \right\}^{\frac{1}{p}}.$$

The unadorned integral denotes integration over the real line $\mathbb{R}$, and a corresponding definition is used for L^∞-norms. The Fourier transform of an integrable function v_0 is defined by

$$\hat{v}_0(\xi) = \frac{1}{\sqrt{2\pi}} \int v_0(x) e^{-ix\xi} \, dx.$$

For a function $v(x, t)$ of two variables, the spatial Fourier transform is denoted by

$$\mathcal{F}_x v(\xi, t) = \frac{1}{\sqrt{2\pi}} \int v(x, t) e^{-ix\xi} \, dx,$$

whereas the notation $\hat{v}(\xi,\tau)$ designates the space–time Fourier transform

$$\hat{v}(\xi,\tau) = \frac{1}{2\pi}\iint v(x,t)e^{-ix\xi}e^{-it\tau}\,dxdt.$$

The Fourier multiplier operators A and Λ are defined by

$$\widehat{Av}(\xi,\tau) = (1+|\xi|)\hat{v}(\xi,\tau)$$

and

$$\widehat{\Lambda v}(\xi,\tau) = (1+|\tau|)\hat{v}(\xi,\tau).$$

Following Foias and Temam [22], the analytic Gevrey norm of order (σ,s) can be written as

$$\|u_0\|_{G^{\sigma,s}} = \|A^s e^{\sigma A}u_0\|_{L^2(\mathbb{R})}.$$

The value of s is of minor significance as the classes considered in this theorem are spaces of analytic functions. In particular, the following lemma shows any number of derivatives of a function to be in G^σ if σ is lowered by an ever-so-small amount.

Proposition 1. *Let ϵ, σ and s be given, such that $0<\epsilon<\sigma$ and $s>0$. Then there exists a constant $c_{s,\epsilon}$ depending on ϵ and s, such that*

$$\|f\|_{G_{\sigma-\epsilon,s}} \le c_{s,\epsilon}\|f\|_{G_\sigma}.$$

Proof. This is a direct consequence of the inequality

$$\sup_{\xi\in\mathbb{R}}\left\{e^{-\epsilon(1+|\xi|)}(1+|\xi|)^s\right\} = c_{s,\epsilon},$$

where $c_{s,\epsilon} = \left(\frac{s}{e}\right)^s\frac{1}{\epsilon^s}$. □

In the following, c will denote a generic constant which may take different values in different places. Dependence on parameters will always be indicated in the text.

If the Eq. (1.3) is linearized, and solutions of the form $e^{i\xi x-i\tau t}$ are sought, then the linear dispersion relation

$$\tau = \sum_{k=1}^{N} b_k\xi(i\xi)^{2k} \equiv \phi(\xi) \tag{2.1}$$

appears. Note here that $\phi(\xi)$ is a polynomial of order $\alpha = 2N+1$. In particular, for the linear initial-value problem

$$\left.\begin{aligned}&\partial_t w + \sum_{k=1}^{N} b_k\partial_x^{2k+1}w = 0,\\ &w(x,0) = w_0(x),\end{aligned}\right\} \tag{2.2}$$

the dispersion relation can be used to give an explicit expression of the solution in terms of the propagator $W(t)$, namely

$$w(x,t) = W(t)w_0 = c\int e^{ix\xi}e^{-it\phi(\xi)}\widehat{w}_0(\xi)\,d\xi. \tag{2.3}$$

In order to prove the local well posedness result, another family of function spaces has to be introduced. For $\sigma>0$, $s\in\mathbb{R}$, and $b\in[-1,1]$, define $X_{\sigma,s,b}$ to be the Banach space equipped with the space–time norm

$$\|v\|^2_{\sigma,s,b} = \iint (1+|\tau+\phi(\xi)|)^{2b}(1+|\xi|)^{2s}e^{2\sigma(1+|\xi|)}|\hat{v}(\xi,\tau)|^2\,d\xi d\tau.$$

The weight in the $X_{\sigma,s,b}$-norm is adapted to the linear part of the Eq. (1.3) through the dispersion relation $\tau=\phi(\xi)$. In particular, we have the following identity connecting the propagator $W(t)$ and the space–time norms $\|\cdot\|_{\sigma,s,b}$.

$$\|W(t)v\|_{\sigma,s,b} = \|A^s e^{\sigma A}\Lambda^b v\|_{L^2(\mathbb{R}^2)}.$$

As already mentioned, the space of continuous functions on the interval $[-T,T]$ with values in $G^{\sigma,s}$ is denoted by $C([-T,T],G^{\sigma,s})$. This space is a Banach space when equipped with the norm

$$|v|_{\mathcal{C}_{T,\sigma,s}} = \sup_{-T\le t\le T}\|v(\cdot,t)\|_{G^{\sigma,s}}.$$

If $b>\frac{1}{2}$, $X_{\sigma,s,b}$ is embedded in $C([-T,T],G^{\sigma,s})$. In fact, the inequality

$$|v|_{\mathcal{C}_{T,\sigma,s}} \le c\|v\|_{\sigma,s,b} \tag{2.4}$$

follows directly from the Sobolev Embedding Theorem. Note that for $\sigma = 0$, the space $X_{\sigma,s,b}$ coincides with the space $X_{s,b}$ introduced by Bourgain [23,24].

We close this section by recalling some linear estimates which hold for these spaces. Let ψ be an infinitely differentiable function on $\mathbb{R}$, such that

$$\psi(t) = \begin{cases} 0, & |t| \geq 2, \\ 1, & |t| \leq 1, \end{cases}$$

and let $\psi_T(t) = \psi(t/T)$.

Lemma 1. *Let $\sigma > 0$, $b > \frac{1}{2}$, and $b - 1 < b' < 0$. Then there is a constant c such that the following estimates hold for any $T \in \mathbb{R}$.*

$$\|\psi(t)\, W(t)v_0(x)\|_{\sigma,s,b} \leq c\, \|v_0\|_{G^{\sigma,s}}, \tag{2.5}$$

$$\left\| \psi_T(t) \int_0^t W(t-s)v(s)\, \mathrm{d}s \right\|_{\sigma,s,b} \leq c\, T^{1-b+b'} \|v\|_{\sigma,s,b'}. \tag{2.6}$$

With the same assumptions as in the lemma, but with $\sigma = 0$, (2.5) was proved in [25], and (2.6) was proved in [26]. These inequalities clearly remain valid for $\sigma > 0$, as one merely has to replace v_0 by $\mathrm{e}^{\sigma A} v_0$ and v by $\mathrm{e}^{\sigma A} v$ in these results.

3. A multilinear estimate

In this section estimates for the nonlinear term in the equation are proved. This result is similar to the estimates of Bourgain and Kenig, Ponce, and Vega [23–25]. Note that in light of Proposition 1, the value of s plays a minor role, and we have not aimed at providing a sharp value for s.

Theorem 2. *Let N be given, let $k, l \leq N$ and $m \leq N - k$. Let $b > \frac{1}{2}$, $b' < -\frac{1}{4}$, and $s \geq (2N+1)b + N$. Let $p \in \mathbb{N}$, and suppose $v, v_1, v_2, \ldots, v_p \in X_{\sigma,s,b}$. Then there exists a constant c depending only on p, σ, s, b, and b' such that*

$$\|\partial_x^k(\partial_x^m v \prod_{i=1}^{p} \partial_x^l v_i)\|_{\sigma,s,b'} \leq c\, \|v\|_{\sigma,s,b} \prod_{i=1}^{p} \|v_i\|_{\sigma,s,b}. \tag{3.1}$$

The proof of Theorem 2 will be established using a number of auxiliary results. To state these results, we first need to introduce some more notation. For $\rho \in \mathbb{R}$, and a suitable f, define F_ρ by way of its Fourier transform $\widehat{F_\rho}$ as

$$\widehat{F}_\rho(\xi,\tau) = \frac{|f(\xi,\tau)|}{(1 + |\tau + \phi(\xi)|)^\rho}.$$

The first lemma is a generalization of a result proved by Bourgain [23].

Lemma 2. *Let $k \leq N$. For $\rho > \frac{1}{4}$, there exists a constant c depending only on ρ such that*

$$\|A^{\frac{k}{2}} F_\rho\|_{L_x^4 L_t^2} \leq c \|f\|_{L_\xi^2 L_\tau^2}. \tag{3.2}$$

Proof. The proof proceeds along the same lines as in [23]. By Plancherel, we have

$$\begin{aligned}
\|A^{\frac{k}{2}} F_\rho\|_{L_t^2}^2 &= \int \left| \iint \frac{(1+|\xi|)^{\frac{k}{2}}\, f(\xi,\tau)}{(1+|\tau+\phi(\xi)|)^\rho} \mathrm{e}^{\mathrm{i}\xi x} \mathrm{d}\xi\, \mathrm{e}^{\mathrm{i}\tau t} \mathrm{d}\tau \right|^2 \mathrm{d}t \\
&= \int \left| \int \frac{(1+|\xi|)^{\frac{k}{2}}\, f(\xi,\tau)}{(1+|\tau+\phi(\xi)|)^\rho} \mathrm{e}^{\mathrm{i}\xi x} \mathrm{d}\xi \right|^2 \mathrm{d}\tau.
\end{aligned}$$

Next using Minkowski's inequality and the Hausdorff–Young Theorem, we obtain

$$\begin{aligned}
\|A^{\frac{k}{2}} F_\rho\|_{L_x^4 L_t^2}^2 &\leq \int \left\| \int \frac{(1+|\xi|)^{\frac{k}{2}}\, f(\xi,\tau)}{(1+|\tau+\phi(\xi)|)^\rho} \mathrm{e}^{\mathrm{i}\xi x} \mathrm{d}\xi \right\|_{L_x^4}^2 \mathrm{d}\tau \\
&\leq \int \left\| \frac{(1+|\xi|)^{\frac{k}{2}}\, f(\xi,\tau)}{(1+|\tau+\phi(\xi)|)^\rho} \right\|_{L_\xi^{\frac{4}{3}}}^2 \mathrm{d}\tau \\
&= \int \left| \int \frac{(1+|\xi|)^{\frac{2}{3}k}\, |f(\xi,\tau)|^{\frac{4}{3}}}{(1+|\tau+\phi(\xi)|)^{\frac{4}{3}\rho}} \mathrm{d}\xi \right|^{\frac{3}{2}} \mathrm{d}\tau.
\end{aligned}$$

Finally, the inner integral is estimated using Hölder's inequality,

$$\int \frac{(1+|\xi|)^{\frac{2}{3}k}\,|f(\xi,\tau)|^{\frac{4}{3}}}{(1+|\tau+\phi(\xi)|)^{\frac{4}{3}\rho}}\mathrm{d}\xi \le \left\| |f(\xi,\tau)|^{\frac{4}{3}} \right\|_{L_\xi^{\frac{3}{2}}} \left\| \frac{(1+|\xi|)^{\frac{2}{3}k}}{(1+|\tau+\phi(\xi)|)^{\frac{4}{3}\rho}} \right\|_{L_\xi^3}$$
$$= \left\{ \int |f(\xi,\tau)|^2 \mathrm{d}\xi \right\}^{\frac{2}{3}} \left\{ \int \frac{(1+|\xi|)^{2k}}{(1+|\tau+\phi(\xi)|)^{4\rho}}\,\mathrm{d}\xi \right\}^{\frac{1}{3}}.$$

Making the change of variables $\lambda = \tau + \phi(\xi)$ in the second factor, the integral becomes

$$\int \frac{(1+|\xi|)^{2k}}{(1+|\lambda|)^{4\rho}} \frac{\mathrm{d}\lambda}{\phi'(\xi)},$$

where it is understood that ξ is now a function of λ. Here, only the highest-order term of $\phi'(\xi)$ plays a role, and the integral is bounded because it was assumed that $k \le N$, so that $2k \le 2N = \alpha - 1$, and ϕ' is a polynomial of order $\alpha - 1$. Thus there appears the inequality

$$\int \frac{(1+|\xi|)^{\frac{2}{3}k}\,|f(\xi,\tau)|^{\frac{4}{3}}}{(1+|\tau+\phi(\xi)|)^{\frac{4}{3}\rho}}\mathrm{d}\xi \le c \left\{ \int |f(\xi,\tau)|^2 \mathrm{d}\xi \right\}^{\frac{2}{3}},$$

which proves the lemma. □

The next two lemmas provide control over the $L_x^\infty L_t^\infty$ and $L_x^2 L_t^\infty$-norms.

Lemma 3. *For $\rho > \frac{1}{2}$, and $s > \frac{1}{2} + k$, there exists a constant c depending only on ρ and s, such that*

$$\|A^{k-s} F_\rho\|_{L_x^\infty L_t^\infty} \le c \|f\|_{L_\xi^2 L_\tau^2}. \tag{3.3}$$

Proof. The proof follows from the Riemann–Lebesgue Lemma and Hölder's inequality.

$$\begin{aligned}
\|A^{k-s} F_\rho\|_{L_x^\infty L_t^\infty} &\le \iint \frac{|f(\xi,\tau)|}{(1+|\tau+\phi(\xi)|)^\rho\,(1+|\xi|)^{s-k}}\,\mathrm{d}\tau\mathrm{d}\xi \\
&\le \int \left\{ \int \frac{|f(\xi,\tau)|^2}{(1+|\xi|)^{2s-2k}}\,\mathrm{d}\tau \right\}^{\frac{1}{2}} \left\{ \int \frac{\mathrm{d}\tau}{(1+|\tau+\phi(\xi)|)^{2\rho}} \right\}^{\frac{1}{2}} \mathrm{d}\xi \\
&\le c \left\{ \iint |f(\xi,\tau)|^2\,\mathrm{d}\tau\mathrm{d}\xi \right\}^{\frac{1}{2}} \left\{ \int \frac{\mathrm{d}\xi}{(1+|\xi|)^{2s-2k}} \right\}^{\frac{1}{2}} \\
&\le c\,\|f\|_{L_\xi^2 L_\tau^2}.
\end{aligned}$$

Here, note that

$$\int \frac{\mathrm{d}\tau}{(1+|\tau+\phi(\xi)|)^{2\rho}} = c < \infty$$

for all values of ξ. □

Lemma 4. *For $\rho > \frac{1}{2}$, and $s \ge (2N+1)\rho + k$, there exists a constant c depending only on ρ and s, such that*

$$\|A^{k-s} F_\rho\|_{L_x^2 L_t^\infty} \le c \|f\|_{L_\xi^2 L_\tau^2}. \tag{3.4}$$

Proof. Let $\alpha = 2N + 1$. Using the Sobolev Embedding Theorem, it can be seen that

$$\begin{aligned}
\|A^{k-s} F_\rho\|_{L_x^2 L_t^\infty} &= \left\| \|A^{k-s} F_\rho\|_{L_t^\infty} \right\|_{L_x^2} \\
&\le c \left\| \|\Lambda^\rho A^{k-s} F_\rho\|_{L_t^2} \right\|_{L_x^2} \\
&= c \left\| \Lambda^\rho A^{k-s} F_\rho \right\|_{L_{x,t}^2} \\
&= c \left\| \frac{(1+|\tau|)^\rho}{(1+|\xi|)^{s-k}\,(1+|\tau+\phi(\xi)|)^\rho} f(\xi,\tau) \right\|_{L_{\xi,\tau}^2}
\end{aligned}$$

$$\leq c \left\| \frac{(1+|\tau|)^{\rho}}{(1+|\xi|)^{s-k}\,(1+|\tau+|\xi|^{\alpha}|)^{\rho}} \right\|_{L^{\infty}_{\xi,\tau}} \|f(\xi,\tau)\|_{L^2_{\xi,\tau}}$$
$$\leq c\, \|f(\xi,\tau)\|_{L^2_{\xi,\tau}}.$$

The fact that

$$\frac{(1+|\tau|)^{\rho}}{(1+|\xi|)^{s-k}\,(1+|\tau+|\xi|^{\alpha}|)^{\rho}}$$

is bounded can readily be verified. □

We are now ready to prove the multilinear estimate in Theorem 2.

Proof of Theorem 2. We only give the proof of the most demanding case, namely when $l = N$ and $k+m = N$. For simplicity of exposition, the proof is first given for the cubic case $p = 2$. The quadratic case $p = 1$ is virtually the same, and the proof for larger values of p can be easily explained. Note that in case $p = 2$, (3.1) can be written more explicitly as

$$\left\|(1+|\xi|)^{s}\,(1+|\tau+\phi(\xi)|)^{b'}\,\mathrm{e}^{\sigma(1+|\xi|)}\left\{\partial_x^k\left(\partial_x^{N-k}v\;\partial_x^N v_1\;\partial_x^N v_2\right)\right\}\hat{\ }(\xi,\tau)\right\|_{L^2_\xi L^2_\tau} \leq c\|v\|_{\sigma,s,b}\|v_1\|_{\sigma,s,b}\|v_2\|_{\sigma,s,b},$$

or

$$\left\|(1+|\xi|)^{s}\,(1+|\tau+\phi(\xi)|)^{b'}\,\mathrm{e}^{\sigma(1+|\xi|)}\,|\xi|^{k}\,\left((\mathrm{i}\xi)^{N-k}\hat{v}*(\mathrm{i}\xi)^{N}\hat{v}_1*(\mathrm{i}\xi)^{N}\hat{v}_2\right)(\xi,\tau)\right\|_{L^2_\xi L^2_\tau}$$
$$\leq c\|v\|_{\sigma,s,b}\|v_1\|_{\sigma,s,b}\|v_2\|_{\sigma,s,b}. \tag{3.5}$$

Now note that if we let

$$f(\xi,\tau) = (1+|\xi|)^{s}\,(1+|\tau+\phi(\xi)|)^{b}\mathrm{e}^{\sigma(1+|\xi|)}\hat{v}(\xi,\tau),$$
$$f_1(\xi,\tau) = (1+|\xi|)^{s}\,(1+|\tau+\phi(\xi)|)^{b}\mathrm{e}^{\sigma(1+|\xi|)}\hat{v}_1(\xi,\tau),$$

and

$$f_2(\xi,\tau) = (1+|\xi|)^{s}\,(1+|\tau+\phi(\xi)|)^{b}\mathrm{e}^{\sigma(1+|\xi|)}\hat{v}_2(\xi,\tau),$$

then (3.5) will be true if it can be shown that

$$\Bigg\|\frac{(1+|\xi|)^{k+s}\ \mathrm{e}^{\sigma(1+|\xi|)}}{(1+|\tau+\phi(\xi)|)^{-b'}}\iiiint \frac{(1+|\xi_1|)^{N-k-s}\,\mathrm{e}^{-\sigma(1+|\xi_1|)}f(\xi_1,\tau_1)}{(1+|\tau_1+\phi(\xi_1)|)^{b}}\,\frac{(1+|\xi-\xi_2|)^{N-s}\,\mathrm{e}^{-\sigma(1+|\xi-\xi_2|)}f_1(\xi-\xi_2,\tau-\tau_2)}{(1+|\tau-\tau_2+\phi(\xi-\xi_2)|)^{b}}$$
$$\times\ \frac{(1+|\xi_2-\xi_1|)^{N-s}\,\mathrm{e}^{-\sigma(1+|\xi_2-\xi_1|)}f_2(\xi_2-\xi_1,\tau_2-\tau_1)}{(1+|\tau_2-\tau_1+\phi(\xi_2-\xi_1)|)^{b}}\,\mathrm{d}\xi_1\mathrm{d}\tau_1\mathrm{d}\xi_2\mathrm{d}\tau_2\Bigg\|_{L^2_\xi L^2_\tau}$$
$$\leq c\|f\|_{L^2_\xi L^2_\tau}\|f_1\|_{L^2_\xi L^2_\tau}\|f_2\|_{L^2_\xi L^2_\tau}.$$

A proof of this estimate can be obtained by duality. Let $d(\xi,\tau)$ be a positive function in $L^2(\mathbb{R}^2)$ with norm $\|d\|_{L^2(\mathbb{R}^2)} = 1$, and let $\mathrm{d}\mu = \mathrm{d}\xi_2\mathrm{d}\tau_2\mathrm{d}\xi_1\mathrm{d}\tau_1\mathrm{d}\xi\mathrm{d}\tau$. Then we have to estimate an integral of the form

$$\int_{\mathbb{R}^6}\frac{d(\xi,\tau)\ (1+|\xi|)^{k+s}\ \mathrm{e}^{\sigma(1+|\xi|)}}{(1+|\tau+\phi(\xi)|)^{-b'}}\ \frac{(1+|\xi_1|)^{N-k-s}f_1(\xi_1,\tau_1)\,\mathrm{e}^{-\sigma(1+|\xi_1|)}}{(1+|\tau_1+\phi(\xi_1)|)^{b}}$$
$$\times\ \frac{(1+|\xi-\xi_2|)^{N-s}f_2(\xi-\xi_2,\tau-\tau_2)\,\mathrm{e}^{-\sigma(1+|\xi-\xi_2|)}}{(1+|\tau-\tau_2+\phi(\xi-\xi_2)|)^{b}}\,\frac{(1+|\xi_2-\xi_1|)^{N-s}f_3(\xi_2-\xi_1,\tau_2-\tau_1)\,\mathrm{e}^{-\sigma(1+|\xi_2-\xi_1|)}}{(1+|\tau_2-\tau_1+\phi(\xi_2-\xi_1)|)^{b}}\,\mathrm{d}\mu.$$

Using the inequality $|\xi| \leq |\xi_1| + |\xi-\xi_2| + |\xi_2-\xi_1|$ on the exponentials, we are left with

$$\int_{\mathbb{R}^6}\frac{d(\xi,\tau)|\xi|^{k+s}}{(1+|\tau+\phi(\xi)|)^{-b'}}\ \frac{(1+|\xi_1|)^{N-k-s}\,|f(\xi_1,\tau_1)|}{(1+|\tau_1+\phi(\xi_1)|)^{b}}\,\frac{(1+|\xi-\xi_2|)^{N-s}\,|f_1(\xi-\xi_2,\tau-\tau_2)|}{(1+|\tau-\tau_2+\phi(\xi-\xi_2)|)^{b}}$$
$$\times\ \frac{(1+|\xi_2-\xi_1|)^{N-s}\,|f_2(\xi_2-\xi_1,\tau_2-\tau_1)|}{(1+|\tau_2-\tau_1+\phi(\xi_2-\xi_1)|)^{b}}\,\mathrm{d}\mu.$$

Now, split the Fourier space into six regions, according to all possible combinations of inequalities such as $|\xi-\xi_2| \leq |\xi_2-\xi_1| \leq |\xi_1|$. In this particular case, the integral can be dominated by the inner product

$$\left\langle A^{\frac{N}{2}}D_{-b'},\, A^{\frac{N}{2}}F_b\, A^{N-s}(F_1)_b\, A^{N-s}(F_2)_b\right\rangle,$$

and the estimate continues as follows.

$$\begin{aligned}\left\langle A^{\frac{N}{2}} D_{-b'}, A^{\frac{N}{2}} F_b\, A^{N-s}(F_1)_b\, A^{N-s} A(F_2)_b \right\rangle &\le c\|A^{\frac{N}{2}} D_{-b'}\|_{L_x^4 L_t^2} \|A^{\frac{N}{2}} F_b\|_{L_x^4 L_t^2} \|A^{N-s}(F_1)_b\|_{L_x^2 L_t^\infty} \|A^{N-s}(F_2)_b\|_{L_x^\infty L_t^\infty} \\ &\le c\|d\|_{L_\xi^2 L_\tau^2} \|f\|_{L_\xi^2 L_\tau^2} \|f_1\|_{L_\xi^2 L_\tau^2} \|f_2\|_{L_\xi^2 L_\tau^2} \\ &= c\|f\|_{L_\xi^2 L_\tau^2} \|f_1\|_{L_\xi^2 L_\tau^2} \|f_2\|_{L_\xi^2 L_\tau^2},\end{aligned}$$

where Lemmas 2–4 were used in the second step. The other cases follow simply by interchanging the roles of F_b, $(F_1)_b$, and $(F_2)_b$.

The proof in the case of higher nonlinearities $p \ge 3$ is virtually identical. The only difference is that we need to split the Fourier space in $p!$ regions. For example, a splitting in which all the combinations are dominated by $|\xi_1|$ will lead to the following estimate.

$$\begin{aligned}&\left\langle A^{\frac{N}{2}} D_{-b'}, A^{\frac{N}{2}} F_b\, A^{N-s}(F_1)_b \prod_{i=2}^{p} (A^{N-s}(F_i)_b) \right\rangle \\ &\le c\|A^{\frac{N}{2}} D_{-b'}\|_{L_x^4 L_t^2} \|A^{\frac{N}{2}} F_b\|_{L_x^4 L_t^2} \|A^{N-s}(F_1)_b\|_{L_x^2 L_t^\infty} \prod_{i=2}^{p} \|A^{N-s}(F_i)_b\|_{L_x^\infty L_t^\infty} \\ &\le c\|d\|_{L_\xi^2 L_\tau^2} \|f\|_{L_\xi^2 L_\tau^2} \|f_1\|_{L_\xi^2 L_\tau^2} \prod_{i=2}^{p} \|f_i\|_{L_\xi^2 L_\tau^2} = c\|f\|_{L_\xi^2 L_\tau^2}\, \Pi_{i=1}^{p} \|f_i\|_{L_\xi^2 L_\tau^2}. \quad \square\end{aligned}$$

4. Existence of a solution

With the estimates provided in the previous two sections, existence of a solution in $X_{\sigma,s,b}$ of the initial-value problem for (1.3) can be proved easily using a contraction argument. Consider the integral operator

$$\Gamma(v) = \psi(t) W(t) u_0 - \sum_{k=0}^{N} \sum_{l=0}^{N} \sum_{m=0}^{N-k} \psi_T(t) \int_0^t W(t-t') \left\{ \partial_x^k \left(\partial_x^m v\, P_{k,l,m}(\partial_x^l v)(t') \right) \right\} \mathrm{d}t'.$$

Let $r = \|u_0\|_{G^{\sigma,s}}$. It will be proved that T can be chosen so that Γ is a contraction in the ball $B(2cr) \subset X_{\sigma,s,b}$ of radius $2cr$ centered at 0.

Lemma 5. *There exists a positive time T, such that the operator Γ is a contraction in the ball $B(2cr)$.*

Proof. First it is proved that Γ is a mapping on $B(2cr)$. Observe that

$$\|\Gamma(v)\|_{\sigma,s,b} \le \|\psi(t) W(t) u_0\|_{\sigma,s,b} + \sum_{k=0}^{N} \sum_{l=0}^{N} \sum_{m=0}^{N-k} \left\| \psi_T(t) \int_0^t W(t-t') \partial_x^k \left(\partial_x^m v\, P_{k,l,m}(\partial_x^l v)(t') \right) \mathrm{d}t' \right\|_{\sigma,s,b}.$$

So using (2.5) and (2.6), and the nonlinear estimates, there appears

$$\begin{aligned}\|\Gamma(v)\|_{\sigma,s,b} &\le c\|u_0\|_{G^{\sigma,s}} + c\, T^{1-b+b'} \sum_{k=0}^{N} \sum_{l=0}^{N} \sum_{m=0}^{N-k} \|\partial_x^k \left(\partial_x^m v\, P_{k,l,m}(\partial_x^l v) \right)\|_{\sigma,s,b'} \\ &\le c\|u_0\|_{G^{\sigma,s}} + c\, T^{1-b+b'} \sum_{k=0}^{N} \sum_{l=0}^{N} \sum_{m=0}^{N-k} \|v\|_{\sigma,s,b}\, P_{k,l,m}(\|v\|_{\sigma,s,b}) \\ &\le cr + c\, T^{1-b+b'} \sum_{k=0}^{N} \sum_{l=0}^{N} \sum_{m=0}^{N-k} 2cr P_{k,l,m}(2cr).\end{aligned}$$

It appears that T can be chosen small enough so that Γ maps $B(2cr)$ to $B(2cr)$. Next, the contraction property will be proved. For two functions v and w in $B(2cr)$, we have the estimate

$$\|\Gamma(v) - \Gamma(w)\|_{\sigma,s,b} \le c\, T^{1-b+b'} \sum_{k=0}^{N} \sum_{l=0}^{N} \sum_{m=0}^{N-k} \left\| \partial_x^k \left(\partial_x^m v\, P_{k,l,m}(\partial_x^l v) \right) - \partial_x^k \left(\partial_x^m w\, P_{k,l,m}(\partial_x^l w) \right) \right\|_{\sigma,s,b'}.$$

First note that if the polynomial $P_{k,l,m}$ is a pure power, say p, then the corresponding part of the nonlinear term can be written as

$$\begin{aligned}\partial_x^k \left(\partial_x^m v\, (\partial_x^l v)^p - \partial_x^m w\, (\partial_x^l w)^p \right) &= \partial_x^k \left(\partial_x^m (v-w)(\partial_x^l v)^p + \partial_x^m w\, \left((\partial_x^l w)^p - (\partial_x^l w)^p \right) \right) \\ &= \partial_x^k \left(\partial_x^m (v-w)(\partial_x^l v)^p + \partial_x^m w\, \partial_x^l (v-w) Q_p(\partial_x^l v, \partial_x^l w) \right),\end{aligned}$$

for some polynomial Q_p of order $p-1$. Thus, the estimate can be continued as

$$\begin{aligned}\|\Gamma(v)-\Gamma(w)\|_{\sigma,s,b} &\le c\,T^{1-b+b'}\sum_{k=0}^{N}\sum_{l=0}^{N}\sum_{m=0}^{N-k}\sum_{p=1}^{K_{k,l,m}} c_{k,l,m,p}\left\|\partial_x^k\left(\partial_x^m(v-w)(\partial_x^l v)^p\right)\right\|_{\sigma,s,b'}\\ &\quad+c\,T^{1-b+b'}\sum_{k=0}^{N}\sum_{l=0}^{N}\sum_{m=0}^{N-k}\sum_{p=1}^{K_{k,l,m}} c_{k,l,m,p}\left\|\partial_x^k\left(\partial_x^m w\,\partial_x^l(v-w)\,Q_p(\partial_x^l v,\partial_x^l w)\right)\right\|_{\sigma,s,b'}\\ &\le c\,T^{1-b+b'}\sum_{k=0}^{N}\sum_{l=0}^{N}\sum_{m=0}^{N-k}\sum_{p=1}^{K_{k,l,m}} c_{k,l,m,p}\|v-w\|_{\sigma,s,b}\|v\|_{\sigma,s,b}^p\\ &\quad+c\,T^{1-b+b'}\sum_{k=0}^{N}\sum_{l=0}^{N}\sum_{m=0}^{N-k}\sum_{p=1}^{K_{k,l,m}} c_{k,l,m,p}\|w\|_{\sigma,s,b}\|v-w\|_{\sigma,s,b}\,Q_p\left(\|v\|_{\sigma,s,b},\|w\|_{\sigma,s,b}\right),\end{aligned}$$

where $K_{k,l,m}$ is the order of the polynomial $P_{k,l,m}$, and $c_{k,l,m,p}$ is the pth coefficient. Since v and w are in $B(2cr)$, it is clear that T can be chosen small enough to obtain the contractive estimate

$$\|\Gamma(v)-\Gamma(w)\|_{\sigma,s,b}\le\frac{1}{2}\|v-w\|_{\sigma,s,b}.\quad\Box$$

Since Γ is a contraction, it follows that Γ has a unique fixed point u in $B(2cr)$. The function u solves the initial-value problem for (1.3). As mentioned in Section 2, $X_{\sigma,s,b}$ is continuously embedded in $C([-T,T],G^{\sigma,s})$ as long as $b>\frac{1}{2}$, so that u also belongs to the space $C([-T,T],G^{\sigma,s})$. However, uniqueness of the solution holds only in the smaller space $X_{\sigma,s,b}$ so far. In the next section, it will be shown how to obtain uniqueness in $C([-T,T],G^{\sigma,s})$.

5. Uniqueness and continuous dependence

Suppose then that there are two solutions u_1 and u_2 of (1.3), both members of the space $C([-T,T],G^{\sigma,s})$, and such that $u_1(x,0)=u_0(x)$ and $u_2(x,0)=u_0(x)$. First, it will be shown that $\psi_{T/2}u_1$ and $\psi_{T/2}u_2$ belong to the space $X_{\sigma-\epsilon,s,b}$, for any ϵ, such that $0<\epsilon<\sigma$.

Lemma 6. *Let $b\in[-1,1]$, $s>\frac{1}{2}$ and $0<\epsilon<\sigma$, and suppose $u\in C([-T,T],G^{\sigma,s})$ satisfies (1.3). Then $\|\psi_{T/2}u\|_{\sigma-\epsilon,s,b}$ is finite.*

Proof. Changing variables in the definition of the norm, it follows immediately that

$$\begin{aligned}\|\psi_{T/2}u\|^2_{\sigma-\epsilon,s,b} &= \int_{-\infty}^{\infty}(1+|\xi|)^{2s}e^{2(\sigma-\epsilon)(1+|\xi|)}\int_{-\infty}^{\infty}\left|\Lambda^b\left(\psi_{T/2}(t)\,e^{-i\phi(\xi)t}\,\mathcal{F}_x u(\xi,t)\right)\right|^2 dt\,d\xi\\ &\le c\int_{-\infty}^{\infty}(1+|\xi|)^{2s}e^{2(\sigma-\epsilon)(1+|\xi|)}\int_{-\infty}^{\infty}\left|\psi_{T/2}(t)\,e^{-i\phi(\xi)t}\,\mathcal{F}_x u(\xi,t)\right|^2 dt\,d\xi\\ &\quad+c\int_{-\infty}^{\infty}(1+|\xi|)^{2s}e^{2(\sigma-\epsilon)(1+|\xi|)}\int_{-\infty}^{\infty}\left|\partial_t\left(\psi_{T/2}(t)\,e^{-i\phi(\xi)t}\,\mathcal{F}_x u(\xi,t)\right)\right|^2 dt\,d\xi.\end{aligned}$$

Differentiating with respect to t, the integrand of the last integral is seen to be

$$\frac{2}{T}\psi'_{T/2}(t)e^{-i\phi(\xi)t}\mathcal{F}_x u(\xi,t)+\psi_{T/2}(t)(-i)\phi(\xi)e^{-i\phi(\xi)t}\mathcal{F}_x u(\xi,t)+\psi_{T/2}(t)e^{-i\phi(\xi)t}\mathcal{F}_x u_t(\xi,t).$$

Thus is appears that $\|\psi_{T/2}u\|^2_{\sigma-\epsilon,s,b}$ is bounded as follows.

$$\begin{aligned}\|\psi_{T/2}u\|^2_{\sigma-\epsilon,s,b} &\le c\int_T^T\int_{-\infty}^{\infty}(1+|\xi|)^{2s}e^{2(\sigma-\epsilon)(1+|\xi|)}|\mathcal{F}_x u(\xi,t)|^2\,d\xi\,dt\\ &\quad+c\,\frac{2}{T}\int_T^T\int_{-\infty}^{\infty}(1+|\xi|)^{2s}e^{2(\sigma-\epsilon)(1+|\xi|)}|\mathcal{F}_x u(\xi,t)|^2\,d\xi\,dt\\ &\quad+c\int_T^T\int_{-\infty}^{\infty}(1+|\xi|)^{2s+2\alpha}e^{2(\sigma-\epsilon)(1+|\xi|)}|\mathcal{F}_x u(\xi,t)|^2\,d\xi\,dt\\ &\quad+c\int_T^T\int_{-\infty}^{\infty}(1+|\xi|)^{2s}e^{2(\sigma-\epsilon)(1+|\xi|)}|\mathcal{F}_x u_t(\xi,t)|^2\,d\xi\,dt\\ &\le 2\,c\,T\sup_{-T\le t\le T}\|u(\cdot,t)\|_{G^{\sigma-\epsilon,s}}+4\,c\sup_{-T\le t\le T}\|u(\cdot,t)\|_{G^{\sigma-\epsilon,s}}\\ &\quad+2\,c\,T\sup_{-T\le t\le T}\|u(\cdot,t)\|_{G^{\sigma-\epsilon,s+\alpha}}+2\,c\,T\sup_{-T\le t\le T}\|u_t(\cdot,t)\|_{G^{\sigma-\epsilon,s}}.\end{aligned}$$

The first two norms are obviously finite, and the third norm is finite by Proposition 1. The last term, containing u_t is finite because u satisfies the Eq. (1.3). □

With Γ as defined before, it appears now that $\psi_{T/2}u_1 = \Gamma(\psi_{T/2}u_1)$, and $\psi_{T/2}u_2 = \Gamma(\psi_{T/2}u_2)$, where the equations hold in the space $X_{\sigma-\epsilon,s,b}$. Thus using the same estimates as in Section 4, we may write

$$\begin{aligned}\|\psi_{T/2}u_1 - \psi_{T/2}u_2\|_{\sigma-\epsilon,s,b} &= \|\Gamma(\psi_{T/2}u_1) - \Gamma(\psi_{T/2}u_2)\|_{\sigma-\epsilon,s,b} \\ &\le \frac{1}{2}\,\|\psi_{T/2}u_1 - \psi_{T/2}u_2\|_{\sigma-\epsilon,s,b}.\end{aligned}$$

This shows that $u_1 \equiv u_2$, at least on the interval $[-T/2, T/2]$. To prove continuous dependence on the initial data, suppose u and $\bar{u}$ are solutions corresponding to initial data u_0 and $\bar{u}_0$. From the uniqueness result, we know that both u and $\bar{u}$ are elements of $X_{\sigma,s,b}$. Following the argument of the proof of Lemma 5, we arrive at

$$\|u - \bar{u}\|_{\sigma,s,b} \le c\,\|u_0 - \bar{u}_0\|_{G^{\sigma,s}} + \frac{1}{2}\,\|u - \bar{u}\|_{\sigma,s,b}.$$

From this inequality, and the inequality (2.4), continuous dependence in $C([0,T], G^{\sigma,s})$ of the solution on the initial data in $G^{\sigma,s}$ is immediate, as shown by the estimate

$$\frac{1}{2}\,|u - \bar{u}|_{\mathcal{C}_{T,\sigma,s}} \le \frac{1}{2}\,c\,\|u - \bar{u}\|_{\sigma,s,b} \le c^2\,\|u_0 - \bar{u}_0\|_{G^{\sigma,s}}.$$

6. Examples and extensions

In this section, we want to indicate a few examples covered by the general equation (1.3). We will concentrate on model equations appearing in the study of water waves, but (1.3) also includes equations originating from other modeling purposes.

Let us briefly recall the rationale behind deriving model equations of the type considered in this paper. The equations are thought to be valid for long-crested waves if the ratio of amplitude to depth is small and approximately equal to the square of the ratio between depth and wavelength. The equations are given in one spatial dimension as there is little variation in the direction transverse to the propagation of the waves. To obtain the approximating equations, one may use an asymptotic expansion of the function u in orders of nonlinearity, or of the energy functional associated with the problem. The last approach, which was pioneered in the work of Craig and Groves [27] and Olver [28], is pleasantly systematic, and can be used to derive model equations of any order. In the scaling corresponding to the situation mentioned above, the equation of first order in [27] is the KdV equation (1.2) with $p = 1$. One may go beyond the first order, to derive the fifth-order model

$$u_t + \frac{3}{2}uu_x + \frac{1}{4}\partial_x(u_x^2) - \frac{1}{2}\partial_x^2(uu_x) + \frac{1}{6}u_{xxx} + \frac{2}{30}u_{xxxxx} = 0.$$

It appears immediately that this equation falls into the class of equations covered by (1.3) when N is equal to 2. At the next order in [27] appears the seventh-order equation

$$\begin{aligned}&u_t + \frac{3}{2}uu_x + \frac{1}{4}\partial_x(u_x^2) - \frac{1}{2}\partial_x^2(uu_x) + \frac{1}{6}u_{xxx} + \frac{2}{30}u_{xxxxx} - \frac{17}{630}u_{xxxxxxx} + \frac{1}{6}\partial_x(u_xu_{xxx}) - \frac{1}{6}\partial_x^2(uu_{xxx}) \\ &\quad - \frac{1}{6}\partial_x^4(uu_x) + \frac{1}{2}\partial_x(uu_x^2) - \frac{1}{2}\partial_x^2(u^2u_x) = 0.\end{aligned}$$

This equation is seen to fall into the framework of (1.3) by choosing $N = 3$, and rewriting the term $\partial_x^4(uu_x)$ as $\partial_x^2(3u_xu_{xx}) + \partial_x^2(uu_{xxx})$. Similar equations were derived by Olver [28], and also by Benney [29].

Two other special situations are the modeling of internal waves, and the case of surface waves with strong surface tension. In the first case, on may study the the so-called extended KdV equation,

$$u_t + uu_x + u^2u_x + u_{xxx} = 0.$$

This equation which in some quarters is known as the Gardner equation is thought to be a better model for internal waves in fluids than the KdV equation [30], and Theorem 1 clearly applies to this equation.

On the other hand, an equation that appears naturally if surface tension plays a role is the Kawahara equation

$$u_t + uu_x + \left(\frac{1}{3} - \beta\right)u_{xxx} + u_{xxxxx} = 0.$$

This equation is a model for surface waves in the situation when the Bond number β is near $\frac{1}{3}$, i.e. for the case when gravity and capillary effects are nearly balanced [31–33].

For examples from outside the realm of water waves, consider the equation

$$u_t + uu_x - u_{xxxxx} = 0,$$

derived in the context of an electromagnetic transmission line by Nagashima in [34], or the Sawada–Kotera equation

$$u_t + 45u^2u_x - 15u_xu_{xx} - 15uu_{xxx} + u_{xxxxx} = 0,$$

which was found from purely theoretical considerations in [35]. Both of these equations are obviously included in the general class (1.3) (writing $uu_{xxx} = \partial_x^2(uu_x) - \frac{2}{3}\partial_x(u_x^2)$ in the latter).

Finally, it should be mentioned that the results in this article immediately generalize to non-local equations which sometimes appear as models for waves at the interface between two fluids. Close examination of the proofs of the multilinear estimates reveals that they only depend on the highest power in the dispersion relation. Thus it is possible to use a dispersion relation of the type

$$\phi(\xi) = \sum_{k=1}^{N} b_k \xi (i\xi)^{2k} + \sum_{k=1}^{N} a_k \xi |\xi|^{2k-1},$$

corresponding to the equation

$$\partial_t u + \sum_{k=0}^{N} \sum_{l=0}^{N} \partial_x^k \left\{ \sum_{m=0}^{N-k} (\partial_x^m u)\, P_{k,l,m}(\partial_x^l u) \right\} + \sum_{k=1}^{N} a_k H \partial_x^{2k} u + \sum_{k=1}^{N} b_k \partial_x^{2k+1} u = 0, \tag{6.1}$$

where H denotes the Hilbert transform, which may be defined by the singular integral

$$Hf(x) = \frac{1}{\pi}\, p.v. \int_{-\infty}^{\infty} \frac{f(x-y)}{y}\, dy.$$

Perhaps more revealing is the Fourier symbol of H which is given by

$$\widehat{Hf}(\xi) = -\mathrm{i}\, \mathrm{sgn}(\xi) \hat{f}(\xi).$$

Thus there appears the final theorem.

Theorem 3. *Let $s > 2N + \frac{1}{2}$. For initial data in $G^{\sigma,s}$, there exists a positive time T, such that the initial-value problem associated to (6.1) is well posed in the space $C([-T, T], G^{\sigma,s})$.*

An example for such an equation is furnished by the Benjamin equation

$$u_t + uu_x - u_{xxx} - Hu_{xx} = 0,$$

which was indicated in [36–38], as a model for long waves at the interface between two fluids in the case of strong capillarity.

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