



A remark on time-analyticity for the Kuramoto–Sivashinsky equation

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1. Introduction

Many applications of time-analyticity of solutions of dissipative evolution equations raise the question of estimating the uniform time-analyticity radius of solutions. For instance, the rate of decay of power spectrum, which is the quantity often measured in experiments, can be estimated in terms of the uniform time-analyticity radius [1]. Also, in [8,10,11], uniform time-analyticity plays a role in establishing algebraic approximations of attractors. Moreover, a rigorous explicit estimate of the time-analyticity domain is important for constructing these approximations since better analyticity estimates lead to better estimates on accuracy.

In this note, we address time-analyticity of solutions of the Kuramoto–Sivashinsky equation $u_t + u_{xxxx} + u_{xx} + uu_x = 0$ with the initial data $u_0 \in L^p(\mathbb{R})$. Since the time-analyticity is not a local property, it is expected that the analyticity domain will depend on the initial data. In Theorem 2.1, we provide this estimate as a function of the initial data and p . In the case $p = 2$, this has been obtained in [6,9,15] by energy estimates on rays in the complex plane. Here we estimate the fundamental solutions corresponding to each ray. In order to optimize the shape of domains, we employ the fact that the fundamental solution of the main linear part is contracting along every ray if $p = 2$, and then use the Riesz interpolation theorem to cover other values of p (cf. Lemma 3.3).

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2. Time-analyticity

Consider the Kuramoto–Sivashinsky equation (KSE)

$$u_t + u_{xxxx} + u_{xx} + uu_x = 0 \tag{2.1}$$

with the initial condition

$$u(x, 0) = u_0(x), \tag{2.2}$$

where $u_0 \in L^p(\mathbb{R})$ with $1 \leq p \leq \infty$. Let X be the set $C_b(\mathbb{R})$ of bounded continuous functions f on \mathbb{R} . It is a Banach space under the norm $\|f\|_X = \|f\|_{L^\infty} = \sup_{x \in \mathbb{R}} |f(x)|$. A function $u: \mathbb{R} \times [0, T) \rightarrow \mathbb{R}$, where $T \in (0, \infty]$, is a solution of the initial value problem (2.1)–(2.2) if

- (i) u is a classical solution of (2.1) for $(x, t) \in \mathbb{R} \times (0, T)$,
- (ii) $u \in L^\infty(\mathbb{R} \times I)$ for every compact interval $I \subseteq (0, T)$, and
- (iii) the initial condition (2.2) is satisfied in the following sense: We have $u(x, t) \rightarrow u_0$ for a.e. $x \in \mathbb{R}$ as $t \rightarrow 0$, and $\limsup_{t \rightarrow 0^+} \|u(\cdot, t)\|_{L^p} < \infty$.

Note that, by local regularity, every solution u satisfies $u \in C^\infty(\mathbb{R} \times (0, T))$. Using the standard iteration argument as in [2,19], it is easy to establish existence of such solutions. Uniqueness is contained in the next theorem.

Theorem 2.1. *Assume $\|u_0\|_{L^p} \leq M$ for some $M \geq 1$. Then there exists a unique solution u of the initial value problem (2.1)–(2.2) on a maximal interval $(0, T_{\max})$ where $T_{\max} \geq T = 1/CM^{4p/(3p-1)}$. Moreover, the function $u(x, t)$ is a restriction of an X -valued analytic function $u(x, t, \tau) + iv(x, t, \tau)$ in the domain*

$$\mathcal{D} = \left\{ t + i\tau \in \mathbb{C}: 0 < t < T, |\tau| \leq \frac{t^{(5p-7)/8(p-1)}}{CM^{p/2(p-1)}} \right\},$$

if

$$p \in [2, \infty], \quad \text{and} \quad \mathcal{D} = \left\{ t + i\tau \in \mathbb{C}: 0 < t < T, |\tau| \leq \frac{t^{(p+1)/4p}}{CM} \right\},$$

if $p \in [1, 2]$. Moreover, for $t + i\tau \in \mathcal{D}$,

$$\|u(\cdot, t, \tau) + iv(\cdot, t, \tau)\|_{L^p} \leq CM \left(1 + \left(\frac{|\tau|}{t} \right)^{(p-2)/2p} \right),$$

if $p \in [2, \infty]$, and

$$\|u(\cdot, t, \tau) + iv(\cdot, t, \tau)\|_{L^2} \leq \frac{CM}{t^{(2-p)/8p}} \left(1 + \left(\frac{|\tau|}{t} \right)^{(2-p)/8p} \right),$$

if $p \in [1, 2]$.

Note that, if $p=2$, the analyticity domain reduces to that obtained in [9] using energy estimates. The exponents associated to the cases $p \in [1, 2]$ and $[2, \infty]$ are different—this might be due to the fact that $p=2$ plays a special role when integration along the rays is concerned (cf. (3.13) below).

Due to our mild assumptions in the definition of solutions, we are also able to conclude time-analyticity of solutions of Eq. (2.1). We say that u solves (2.1) for $t \in (0, T)$, where $T > 0$, if (i) and (ii) hold.

Theorem 2.2. *Let u be a solution of (2.1) for $t \in (0, T)$ such that $\|u(\cdot, t)\|_{L^\infty} \leq M$ for $t \in (0, T)$ where $M \geq 1$. Then $u(x, t)$ is a restriction of an X -valued analytic function $u(x, t, \tau) + iv(x, t, \tau)$ in the domain*

$$\mathcal{D} = \left\{ t + i\tau \in \mathbb{C}: 0 < t < \frac{1}{CM^{4/3}}, t < T, |\tau| \leq \frac{t^{5/8}}{CM^{1/2}} \right\} \cup \left\{ t + i\tau \in \mathbb{C}: \frac{1}{CM^{4/3}} < t < T, |\tau| \leq \frac{1}{CM^{4/3}} \right\}$$

for some positive universal constants C .

For an estimate of the domain of space analyticity, see [13,18].

Assuming Theorem 2.1, we shall now prove Theorem 2.2 and delay the proof of Theorem 2.1 to the next section.

Proof. By our definition of solutions, $v = u$ is a solution of the initial value problem

$$v_t + v_{xxxx} + v_{xx} + vv_x = 0, \quad t \in [\varepsilon, T)$$

$$v(\cdot, \varepsilon) = u(\cdot, \varepsilon).$$

Therefore, Theorem 2.1 applies with $p = \infty$, and we get the description of the domain provided $t \leq t_0 = 1/CM^{4/3}$. If $t \geq t_0$, we apply the preceding to the function $v(\tau) = u(\tau + t - t_0/2)$. \square

3. Proof of Theorem 2.1

We start by uniqueness of solutions of the initial value problem (2.1) and (2.2). The proof uses some ideas from [7]. Let $Bu = u_{xxxx} + u_{xx}$, denote by $\Gamma(x, t) = (1/2\pi) \int e^{-\zeta^4 t + \zeta^2 t + i\zeta x} d\zeta$ the fundamental solution of the equation $v_t + Bv = 0$, and let e^{-tB} be the corresponding solution operator.

Lemma 3.1. *Assume that u is a solution of the initial value problem (2.1) and (2.2), where $u_0 \in L^p(\mathbb{R})$ with $1 \leq p \leq \infty$. Then we have $t^{(2-p)/8p} u(t) \in L^\infty((0, T'), L^2(\mathbb{R}))$ for every $T' \in (0, T)$ if $p \in [1, 2]$ and*

$$u(t) = e^{-tB} u_0 - \frac{1}{2} \int_0^t \partial_x e^{-(t-s)B} u(s)^2 ds \tag{3.1}$$

holds for all $t \in (0, T)$, where $u(t) = u(\cdot, t)$.

Proof. Assume first $u \in C^\infty(S_T) \cap L^\infty(S_T)$ where $S_T = \mathbb{R} \times [0, T]$. Integration by parts leads to

$$\begin{aligned} & \int \int_{S_T} u(w, s) \left(-\phi_t(w, s) + \phi_{xxxx}(w, s) + \phi_{xx}(w, s) - \frac{1}{2}u(w, s)\phi_x(w, s) \right) dw ds \\ &= \int_{\mathbb{R}} u(w, 0)\phi(w, 0) dw \end{aligned} \tag{3.2}$$

for all $\phi \in \mathcal{D}_T = \{\theta \in \mathcal{S}(\mathbb{R} \times \mathbb{R}) : \text{supp } \theta \subseteq \mathbb{R} \times (-1, T)\}$. Choose a function $\psi \in C^\infty(\mathbb{R})$ such that $0 \leq \psi \leq 1$, $\psi' \geq 0$, $\psi(t) = 0$ for $t \leq 1$, and $\psi(t) = 1$ for $t \geq 2$. Fix $T_1, T_2 > 0$ such that $0 < T_1 < T_2 < T$, let $(x, t) \in \tilde{S} = \mathbb{R} \times (T_1, T_2)$, and let $\varepsilon \in (0, T_1/2)$. Now, consider $\phi(w, s) = \psi(s + 2)\psi((t - s)/\varepsilon)\Gamma(x - w, t - s)$ for $(w, s) \in \mathbb{R}^2$. Then $\phi(w, s) = 0$ if $s \leq -1$ or $s \geq t - \varepsilon$. Using this ϕ as a test function in (3.2), we get

$$\begin{aligned} & - \int \int u(w, s)\psi'(s + 2)\psi((t - s)/\varepsilon)\Gamma(x - w, t - s) dw ds \\ &+ \frac{1}{\varepsilon} \int \int u(w, s)\psi(s + 2)\psi'((t - s)/\varepsilon)\Gamma(x - w, t - s) dw ds \\ &+ \frac{1}{2} \int \int u(w, s)^2\psi(s + 2)\psi((t - s)/\varepsilon)\Gamma_x(x - w, t - s) dw ds \\ &= \int u(w, 0)\Gamma(x - w, t) dw, \end{aligned}$$

where we used $\Gamma_t(w, s) + \Gamma_{xxxx}(w, s) + \Gamma_{xx}(w, s) = 0$ for $s \geq 2\varepsilon$. Denote by $I_j(x, t)$ the j th term of the left-hand side in the previous equality. Clearly, $I_1 = 0$. As $\varepsilon \rightarrow 0$, the third term $I_3(x, t)$ converges to $\frac{1}{2} \int_0^t \int u(w, s)^2 \Gamma_x(x - w, t - s) dw ds$ in $L^\infty(\tilde{S})$. It remains to be shown that $I_2(x, t)$ converges to $u(x, t)$ for almost every $(x, t) \in \tilde{S}$ as $\varepsilon \rightarrow 0$. By a change of variables, we have

$$\begin{aligned} I_2(x, t) - u(x, t) &= \int_0^\infty \int \psi'(s)(u(x - \varepsilon^{1/4}w, t - \varepsilon s) - u(x, t)) \\ &\quad \varepsilon^{1/4} \Gamma(\varepsilon^{1/4}w, \varepsilon s) dw ds, \end{aligned}$$

where we used $\int_{\mathbb{R}} \Gamma(w, t) dw = 1$ for every $t > 0$. As $\varepsilon \rightarrow 0$, the above integral converges to 0 by (i) and the dominated convergence theorem.

Now, we turn to the general case. By (iii), $u \in L^{p, \infty}(S_{T_0}) = L^\infty((0, T_0), L^p(\mathbb{R}))$ for some $T_0 \in (0, T_2)$, while, by (iii), $u \in L^\infty(\mathbb{R} \times (T_0/2, T_2))$.

By the first part of the proof,

$$u(x, t) = \int \Gamma(x - w, t - \varepsilon)u(w, \varepsilon) dw - \frac{1}{2} \int_\varepsilon^t \int \Gamma_x(x - w, t - s)u(w, s)^2 dw ds \tag{3.3}$$

for every $(x, t) \in \mathbb{R} \times (\varepsilon, T)$. If $p \geq 2$, we obtain the conclusion by sending $\varepsilon \rightarrow 0$. If, on the other hand, $p \in [1, 2]$, we get from (3.3) that

$$\sup_{\varepsilon \leq t \leq T_0} (t - \varepsilon)^{(2-p)/4p} \int u(x, t)^2 dx \leq M, \tag{3.4}$$

for every $\varepsilon \in (0, T_0)$ where M is independent of ε . We then get (3.4) with $\varepsilon = 0$ by Fatou’s lemma. By sending $\varepsilon \rightarrow 0$ in (3.3), we get (3.1). It remains to be shown that (3.4) holds with $\varepsilon = 0$ and with T_0 replaced by T_2 . For this, note that $u(T_0/2) \in L^2(\mathbb{R})$ and then use uniqueness of mild solutions with initial data in $L^2(\mathbb{R})$. \square

With the help of the above lemma, it is possible to establish local well-posedness of solutions in L^p . The existence is established by a contraction mapping principle in suitable Banach spaces, while uniqueness and continuity with respect to initial data follow from Lemma 3.1. By regularity of solutions and by integration by parts, we obtain an energy inequality $(d/dt)\|u\|_{L^2}^2 \leq (1/2)\|u\|_{L^2}^2$ which implies global existence for $p = 2$. This in turn also implies global existence of solutions for $p \in [1, 2]$ as well (cf. Proof of Theorem 2.1 below). Global existence of solutions for $p \in (2, \infty)$ can be shown by the method of splitting of initial data and the equation due to Calderón [3,4]. Namely, we have the following statement.

Proposition 3.2. *Let $u_0 \in L^p(\mathbb{R}^n)$ where $p \in [1, \infty)$. Then there exists a solution of (2.1)–(2.2) defined on $[0, \infty)$.*

Proof. The only interesting case is $p \in (2, \infty)$. Due to uniqueness, it is sufficient to construct a solution on $[0, T_0]$ for any given $T_0 > 0$. For an $\varepsilon > 0$, write $u_0 = v_0 + w_0$, where $\|v_0\|_{L^p} < \varepsilon$ and $w_0 \in L^2(\mathbb{R}^n)$. The number ε is chosen so that there exists a solution $v \in C([0, T_0], L^p(\mathbb{R}^n))$ of

$$v_t + v_{xxxx} + v_{xx} + vv_x = 0,$$

$$v(\cdot, 0) = v_0.$$

Moreover, it can be chosen so that $\|v(t)\|_{L^p} \leq 2\varepsilon$ and $\|v_x(t)\|_{L^p} \leq C\varepsilon/t^{(p+1)/4p}$ for $t \in (0, T_0)$. Now, consider

$$w_t + w_{xxxx} + w_{xx} + vw_x + vw_x + ww_x = 0,$$

$$w(\cdot, 0) = w_0. \tag{3.5}$$

or in its integral form

$$w(t) = e^{-tB}w_0 - \int_0^t \partial_x e^{-(t-s)B}v(s)w(s) \, ds - \frac{1}{2} \int_0^t \partial_x e^{-(t-s)B}v(s)^2 \, ds.$$

From Eq. (3.5) we derive $(d/dt)\|w\|_{L^2}^2 \leq (\|v_x\|_{L^\infty} + 1/2)\|w\|_{L^2}^2$. This easily implies $w \in C([0, T_0], L^2(\mathbb{R}^n))$. Also, we have

$$\|w(t)\|_{L^p} \leq C\varepsilon \int_0^t \frac{\|w(s)\|_{L^2}^2 \, ds}{(t-s)^{3/8}} + C \int_0^t \frac{\|w(s)\|_{L^2}^2 \, ds}{(t-s)^{(2p-1)/4p}} + \frac{C\|w_0\|_{L^2}}{t^{(p-2)/8p}},$$

which provides a bound for $\|w(t)\|_{L^p}$ on compact subsets of $(0, T_0)$. The uniform bound in a small neighborhood of 0 can be derived easily from $w_0 = u_0 - v_0 \in L^p(\mathbb{R}^n)$. Finally, note that $u = v + w$ is a solution of (2.1) and (2.2). \square

We add that for periodic locally L^2 initial data, one can show $\lim_{t \rightarrow \infty} \leq C(L)$ where L is the space period [5,12,16].

Now, we describe the procedure for constructing solutions and deducing analyticity of solutions. Let $u^{(0)} \equiv 0$, and let $u^{(n)}$, where $n \in \mathbb{N}$, be the solution of the initial value problem

$$u_t^{(n)} + u_{xxxx}^{(n)} + u_{xx}^{(n)} = -u^{(n-1)}u_x^{(n)}, \quad t > 0,$$

$$u^{(n)}(x, 0) = u_0(x), \quad x \in \mathbb{R},$$

where $u_0 \in L^p(\mathbb{R})$. It is well-known that, for every $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $u^{(n)}(\cdot, t)$ is a restriction of an X -valued analytic function $u^{(n)}(\cdot, t \cos \theta, t \sin \theta) + iv^{(n)}(\cdot, t \cos \theta, t \sin \theta)$ in the domain $\{te^{i\theta} \in \mathbb{C} : t \cos \theta > 0\}$ (cf. [14, Chapter 7] for the case of the heat equation). Introduce

$$U_\theta^{(n)}(x, t) = u^{(n)}(x, t \cos \theta, t \sin \theta)$$

$$V_\theta^{(n)}(x, t) = v^{(n)}(x, t \cos \theta, t \sin \theta)$$

for $n \in \mathbb{N}_0$, $x \in \mathbb{R}$, $t > 0$, and $\theta \in (-\pi/2, \pi/2)$. Omitting the subscript θ , we obtain

$$\partial_t \begin{pmatrix} U^{(n)} \\ V^{(n)} \end{pmatrix} + A \begin{pmatrix} U_{xxxx}^{(n)} + U_{xx}^{(n)} \\ V_{xxxx}^{(n)} + V_{xx}^{(n)} \end{pmatrix} = -A \begin{pmatrix} U^{(n-1)}U_x^{(n-1)} - V^{(n-1)}V_x^{(n-1)} \\ U^{(n-1)}V_x^{(n-1)} + V^{(n-1)}U_x^{(n-1)} \end{pmatrix}, \tag{3.6}$$

where

$$A = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix},$$

while the initial condition reads as

$$\begin{pmatrix} U^{(n)}(\cdot, 0) \\ V^{(n)}(\cdot, 0) \end{pmatrix} = \begin{pmatrix} u_0 \\ 0 \end{pmatrix}. \tag{3.7}$$

The fundamental matrix of the principal part of (3.6) is

$$\Gamma_\theta(x, t) = \begin{pmatrix} \Gamma_1(x, t) & -\Gamma_2(x, t) \\ \Gamma_2(x, t) & \Gamma_1(x, t) \end{pmatrix}$$

where $\Gamma_1(x, t) = (1/2\pi) \int \exp(-(\zeta^4 - \zeta^2)t \cos \theta) \cos(-(\zeta^4 - \zeta^2)t \sin \theta) e^{i\zeta x} d\zeta$ and $\Gamma_2(x, t) = (1/2\pi) \int \exp(-(\zeta^4 - \zeta^2)t \cos \theta) \sin(-(\zeta^4 - \zeta^2)t \sin \theta) e^{i\zeta x} d\zeta$. From (3.6) and (3.7), we obtain

$$\begin{pmatrix} U^{(n)}(t) \\ V^{(n)}(t) \end{pmatrix} = e^{-tB_\theta} \begin{pmatrix} u_0(y) \\ 0 \end{pmatrix} - \frac{1}{2} \int_0^t \partial_x e^{-(t-s)B_\theta} \begin{pmatrix} U^{(n-1)}(s)^2 - V^{(n-1)}(s)^2 \\ 2U^{(n-1)}(s)V^{(n-1)}(s) \end{pmatrix} ds, \tag{3.8}$$

where e^{-tB_θ} is the solution operator corresponding to the left-hand side of (3.6). Also, denote by e^{-tB_j} the convolution with Γ_j for $j = 1, 2$. The following is the main lemma for establishing the domain of analyticity.

Lemma 3.3. *There exists a constant $C_m > 0$, which depends only on $m \in \mathbb{N}_0$, such that*

$$\|\hat{\partial}_x^m \Gamma_j(\cdot, t)\|_{L^p} \leq \frac{C_m e^{t \cos \theta}}{t^{(mp+p-1)/4p} (\cos \theta)^{(mp-p+3)/4p}} \tag{3.9}$$

for all $m \in \mathbb{N}_0$, $t > 0$, $j = 1, 2$, and $p \in [1, 2]$. Also, for $t > 0$ and $j = 1, 2$, we have

$$\|e^{-tB_j} u_0\|_{L^p} \leq \frac{C e^{t \cos \theta}}{(\cos \theta)^{|p-2|/2p}} \|u_0\|_{L^p} \tag{3.10}$$

provided $1 \leq p \leq \infty$, and

$$\|e^{-tB_j} u_0\|_{L^2} \leq \frac{C e^{t \cos \theta}}{t^{(2-p)/8p} (\cos \theta)^{(2-p)/8p}} \|u_0\|_{L^p} \tag{3.11}$$

provided $1 \leq p \leq 2$.

Expressions $e^{t \cos \theta}$ in (3.9)–(3.11) may be reduced to $e^{t(1/4+\varepsilon) \cos \theta}$ for every $\varepsilon > 0$ using arguments below.

Proof. We start with the inequality

$$\|f\|_{L^p} \leq C_p \|f\|_{L^2}^{(3p-2)/2p} \|xf\|_{L^2}^{(2-p)/2p} \tag{3.12}$$

valid if $p \in [1, 2]$ and if the right-hand side is finite. Namely, if $f \in \mathcal{S}$ is not identically zero, if $a > 0$, and if $p \in [1, 2]$, then

$$\begin{aligned} \|f\|_{L^p}^p &\leq \|(x^2 + a^2)^{1/2} f(x)\|_{L^2}^p \|(x^2 + a^2)^{-1}\|_{L^{p/(2-p)}}^{p/2} \\ &= \left(\int x^2 f(x)^2 dx + a^2 \int f(x)^2 dx \right)^{p/2} \|(x^2 + a^2)^{-1}\|_{L^{p/(2-p)}}^{p/2}. \end{aligned}$$

Choosing

$$a = \|h\|_{L^2} \|xf\|_{L^2} / \|xh\|_{L^2} \|f\|_{L^2}$$

where

$$h(x) = (x^2 + 1)^{-1/(2-p)}$$

gives (3.12) with a sharp constant

$$C_p = \frac{\|h\|_{L^p}}{\|h\|_{L^2}^{(3p-2)/2p} \|xh\|_{L^2}^{(2-p)/2p}}.$$

Using Fourier transform, we obtain

$$\|\hat{\partial}_x^m \Gamma_j(\cdot, t)\|_{L^2(\mathbb{R})} \leq \frac{C_m e^{t \cos \theta}}{t^{(2m+1)/8} (\cos \theta)^{(2m+1)/8}}$$

and

$$\|x \hat{\partial}_x^m \Gamma_j(\cdot, t)\|_{L^2(\mathbb{R})} \leq \frac{C_m e^{t \cos \theta}}{t^{(2m-1)/8} (\cos \theta)^{(2m+7)/8}}$$

for all $m \in \mathbb{N}_0$, $j \in \{1, 2\}$, and $t > 0$, and (3.9) is proved.

Consider the solution of

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} + A \begin{pmatrix} u_{xxxx} + u_{xx} \\ v_{xxxx} + v_{xx} \end{pmatrix} = 0$$

for which $u(0) = u_1 \in L^2(\mathbb{R})$ and $v(0) = v_1 \in L^2(\mathbb{R})$. We get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|u\|_{L^2}^2 + \|v\|_{L^2}^2) + \cos \theta (\|u_{xx}\|_{L^2}^2 + \|v_{xx}\|_{L^2}^2) \\ & \leq \cos \theta (\|u_x\|_{L^2}^2 + \|v_x\|_{L^2}^2), \end{aligned} \tag{3.13}$$

hence $\|u(t)\|_{L^2}^2 + \|v(t)\|_{L^2}^2 \leq e^{t \cos \theta} (\|u_1\|_{L^2}^2 + \|v_1\|_{L^2}^2)$. Therefore, for a fixed $j \in \{1, 2\}$, we get (3.10) for $p=2$. On the other hand, (3.9) implies (3.10) for $p=\infty$. Therefore, by the Riesz interpolation theorem (cf. [17, p. 179]), we get (3.10) for $p \in [2, \infty]$; the case $p \in [1, 2]$ then follows by duality. Also, by (3.9), we get (3.11) with $p=1$; other values of $p \in [1, 2]$ are again covered by the Riesz interpolation theorem. \square

Proof of Theorem 2.1. First, we start with the case $p \in [2, \infty)$. Denoting $a_n = \|U^{(n)}\|_{L^{p,\infty}(S_T)} + \|V^{(n)}\|_{L^{p,\infty}(S_T)}$ we get for every $T > 0$ and $n \geq 2$

$$a_n \leq \frac{C_0 e^{T \cos \theta}}{(\cos \theta)^{(p-2)/2p}} \|u_0\|_{L^p} + \frac{C e^{T \cos \theta} T^{(3p-1)/4p}}{(\cos \theta)^{(3p-3)/4p}} a_{n-1}^2.$$

If

$$T \leq \frac{1}{C} \min \left\{ \frac{1}{\cos \theta}, \frac{(\cos \theta)^{(5p-7)/(3p-1)}}{\|u_0\|_{L^p}^{4p/(3p-1)}} \right\}$$

with sufficiently large constant C , we deduce $a_n \leq 2C_0 \|u_0\|_{L^p} / (\cos \theta)^{(p-2)/2p}$ for every $n \geq 1$ and thus conclude

$$\|u^{(n)}(\cdot, t \cos \theta, t \sin \theta)\|_{L^p} + \|v^{(n)}(\cdot, t \cos \theta, t \sin \theta)\|_{L^p} \leq \frac{C \|u_0\|_{L^p}}{(\cos \theta)^{(p-2)/2p}}$$

provided

$$te^{i\theta} \in \mathcal{D} = \left\{ se^{i\phi} \in \mathbb{C}: 0 < s \leq \frac{1}{C} \min \left\{ \frac{1}{\cos \phi}, \frac{(\cos \phi)^{(5p-7)/(3p-1)}}{\|u_0\|_{L^p}^{4p/(3p-1)}} \right\} \right\}. \tag{3.14}$$

On the other hand, standard techniques [19] show that, if C in (3.14) is large enough, $u^{(n)}$ converges in $\mathcal{D} \cap (0, \infty) = (0, C^{-1} \min\{1, \|u_0\|_{L^p}^{-4p/(3p-1)}\})$ to a solution of the KSE. Employing the vector version of the classical Vitali’s theorem, we obtain the assertions.

Now, let $p \in [1, 2]$. Using (3.11) and (3.8), we get for every $t > 0$ that

$$\begin{aligned} & \|U^{(n)}(\cdot, t)\|_{L^2} + \|V^{(n)}(\cdot, t)\|_{L^2} \\ & \leq \frac{C e^{t \cos \theta}}{t^{(2-p)/8p} (\cos \theta)^{(2-p)/8p}} \|u_0\|_{L^p} + \frac{C}{(\cos \theta)^{3/8}} \int_0^t \frac{e^{(t-s) \cos \theta}}{(t-s)^{3/8}} \\ & \quad (\|U^{(n-1)}(\cdot, s)\|_{L^2} + \|V^{(n-1)}(\cdot, s)\|_{L^2})^2 ds. \end{aligned}$$

Denoting $\phi^{(n)}(x, t) = t^{(2-p)/8p} U^{(n)}(x, t)$ and $\psi^{(n)}(x, t) = t^{(2-p)/8p} V^{(n)}(x, t)$ for $x \in \mathbb{R}$ and $t > 0$ and assuming $t \cos \theta \leq 1$, we get

$$\begin{aligned} & \|\phi^{(n)}(\cdot, t)\|_{L^2} + \|\psi^{(n)}(\cdot, t)\|_{L^2} \\ & \leq \frac{C}{(\cos \theta)^{(2-p)/8p}} \|u_0\|_{L^p} + \frac{Ct^{(2-p)/8p}}{(\cos \theta)^{3/8}} \int_0^t \frac{1}{(t-s)^{3/8} s^{(2-p)/4p}} \\ & \quad \times (\|\phi^{(n-1)}(\cdot, s)\|_{L^2} + \|\psi^{(n-1)}(\cdot, s)\|_{L^2})^2 ds \end{aligned}$$

whence, denoting $a_n = \|\phi^{(n)}\|_{L^2, \infty(S_T)} + \|\psi^{(n)}\|_{L^2, \infty(S_T)}$,

$$a_n \leq \frac{C}{(\cos \theta)^{(2-p)/8p}} \|u_0\|_{L^p} + \frac{CT^{(3p-1)/4p}}{(\cos \theta)^{3/8}} a_{n-1}^2$$

provided $0 < T < 1/\cos \theta$. Hence,

$$a_n \leq \frac{C}{(\cos \theta)^{(2-p)/8p}} \|u_0\|_{L^p} \quad \text{for every } n \geq 1$$

provided

$$0 \leq T \leq \frac{1}{C} \min \left\{ \frac{1}{\cos \theta}, \frac{(\cos \theta)^{(p+1)/(3p-1)}}{\|u_0\|_{L^p}^{4p/(3p-1)}} \right\}.$$

We get

$$\|U^{(n)}(\cdot, t)\|_{L^2} + \|V^{(n)}(\cdot, t)\|_{L^2} \leq \frac{C}{t^{(2-p)/8p} (\cos \theta)^{(2-p)/8p}} \|u_0\|_{L^p}$$

for $t \in (0, T)$. Using the weighted space of Weissler

$$W(T) = \left\{ u: \|u\|_{W(T)} = \sup_{0 < t < T} \|u(\cdot, t)\|_{L^p} + \sup_{0 < t < T} t^{(2-p)/8p} \|u(\cdot, t)\|_{L^2} < \infty \right\}$$

(cf. [2,19]), we can show that our sequence converges to a solution of the KSE on the interval $(0, C^{-1} \min\{1, \|u_0\|_{L^p}^{-4p/(3p-1)}\})$. The assertions then follow similarly as above. \square

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