Mathematical Models and Methods in Applied Sciences
Vol. 13, No. 3 (2003) 345–360

SPATIAL ANALYTICITY PROPERTIES OF NONLINEAR WAVES

JERRY L. BONA

Department of Mathematics, Statistics and Computer Science,
University of Illinois at Chicago, Chicago, IL 60607, USA
bona@math.uic.edu

ZORAN GRUJIĆ

Department of Mathematics, University of Virginia,
Charlottesville, VA 22904, USA
zg7c@virginia.edu

Received 25 June 2002
Communicated by F. Brezzi

Dedicated to Jim Douglas, Jr., colleague, mentor and friend, on the occasion of his 75th birthday

In this paper, we study spatial analyticity properties of two classes of equations modeling unidirectional waves in nonlinear, dispersive media, namely KdV-type equations and BBM-type equations. The commentary begins with KdV-type equations and the observation that, for a class of such equations, boundedness of a solution suffices to maintain analyticity and so loss of analyticity detects loss of L_∞-regularity. For a larger class of KdV-type equations, the same conclusion is valid provided that L_∞-boundedness of a solution is replaced by W^1_∞-boundedness. It is also shown that these nonlinear dispersive wave equations are amenable to Gevrey-class analysis based on the boundedness of a Sobolev norm. This analysis yields an *explicit* lower bound on the possible rate of decrease in time of the uniform radius of analyticity of a solution in terms of the assumed Sobolev bound and the Gevrey-norm of the initial data. Attention is then shifted to BBM-type equations. It is shown that, regardless of the strength of the nonlinearity, a solution starting in a Gevrey space remains in this class for all time. Moreover, a lower bound on the possible rate of decrease in time of the uniform analyticity radius has temporal asymptotics that are *independent* of the degree of the nonlinearity, and so apparently determined in the main by the dispersion.

Keywords: Korteweg-de Vries-type equations; BBM-type equations; analytic solutions of nonlinear wave equations; Gevrey-class regularity; temporal asymptotics.

AMS Subject Classification: 30H05, 35Q53, 35B65, 76B15

1. Introduction

This paper is concerned with evolution equations modeling unidirectional propagation of nonlinear waves in dispersive media. In the first part, we study Korteweg-de Vries-type (KdV-type henceforth) equations, and in the second part regularized long-wave or BBM-type equations. The KdV-type equations considered here have the form

$$u_t + G(u)u_x - Lu_x = 0\,, \tag{1}$$

while the BBM-type equations are

$$u_t + G(u)u_x + Lu_t = 0\,. \tag{2}$$

Here, the dependent variable u is a function of two real variables x and t, G is a function that is analytic at least in a neighborhood of zero in $\mathbb{C}$, but real-valued on the real axis, and L is a Fourier multiplier operator. The function G, which in applications is usually a polynomial, reflects nonlinear effects while the Fourier multiplier operator L given by

$$\widehat{Lv}(\xi) = \alpha(\xi)\hat{v}(\xi) \tag{3}$$

has a symbol α that determines and is determined by the linearized dispersion relation arising in the physical situation under consideration. (Here and below, a circumflex over a function connotes that function's Fourier transform with respect to the spatial variable x.) For a discussion of the modeling issues underlying these two classes and both formal and rigorous comparison between them, we may safely refer to Refs. 1, 8 and 9 and the references therein.

Interest is focused upon the initial-value problem wherein a wave profile

$$u(x,0) = \varphi(x)\,, \quad \text{for } x \in \mathbb{R}\,, \tag{4}$$

is specified at $t = 0$, say, and the further evolution is then determined by (1) or (2) with u starting at φ as in (4). While there might be some objections to posing a problem on an infinitely extended domain, there is a considerable mathematical advantage gained by the absence of lateral boundaries. The description thereby obtained is potentially valid far from the interference of solid boundaries.

The present work is focused upon analyticity properties of solutions of (1) or (2) with respect to the spatial variable x. Starting from initial data φ that has an analytic extension to an open, complex strip

$$S_\tau = \{z \in \mathbb{C} : |\Im(z)| < \tau\} \tag{5}$$

for some $\tau > 0$, it has been established that a solution of (1) with φ as initial data remains analytic in a similar strip at least for a short time, although the width of the strip may decrease with elapsed time (see e.g. Ref. 18). (Local-in-time well-posedness for (1) with $L = -\partial_x^2$ and $G(z) = z^p$, the so-called generalized KdV-equation, in analytic spaces on a fixed *closed* strip has been recently obtained in

Ref. 17.) Our purpose here is to delve a little more deeply into this situation. In particular, is *local* persistence of analyticity all that one can hope for? Put another way, is it possible to lose analyticity in finite time? This question may be related to singularity formation which has come to the fore as an interesting issue lately (see Refs. 3, 6, 7 and 23).

In the first two sections, attention is given to finding function classes X of low spatial regularity having the property that X-regularity suffices to maintain analyticity. The first result of this type was obtained by Kato and Masuda[20] for the generalized KdV-equation utilizing an abstract theorem about certain parametric Lyapunov families in Banach spaces. (By generalizing slightly their proof, one determines that their result holds for the general model (1) as well.) Let $H^\infty = \cap_{s\geq 0} H^s$ where, for $s \geq 0$, H^s is the standard, L_2-based Sobolev class of order s. They showed that as long as the solution remains in $X = H^\infty$, then the solution continues to be analytic in a strip of the type exhibited in (5). In Sec. 2, we observe that combining the result in Ref. 20 with a criterion for finite-time blow-up of solutions of the generalized KdV-equation derived by Albert, Bona and Felland,[2] one obtains the stronger result that, as long as the solution $u(\cdot, t)$ lies in $X = L_\infty$, it remains analytic in a strip. In Sec. 3, it is demonstrated that a solution of (1) will retain H^k-regularity, for any $k \geq 1$ provided that both the solution and its derivative are bounded in L_∞. Utilizing this result in place of the criterion of Ref. 2, we deduce that for more general models of the form (1), boundedness in $X = W^1_\infty$ implies the continuation of analyticity. The last section about KdV-type models is focused on the problem of deriving an explicit lower bound on the possible rate of decrease in time of the uniform radius of analyticity τ starting from analytic initial data, and assuming the boundedness of a suitable Sobolev norm. The estimates presented in Ref. 20 do not provide explicit bounds on τ as a function of t. Here, we show that (1) is well-suited to a Gevrey-class approach (see Ref. 15 where the method was introduced, and also Refs. 13, 14, 16, 22 and 24). Such an analysis yields a simple, explicit, lower bound on the decrease of τ in terms of the analytic Gevrey-norm of the initial data and the assumed Sobolev bound.

In the second part of the paper, consideration is turned to BBM-type equations as in (2). Global-in-time well-posedness of (2) (say with $G(z) = z^p$) in H^s for suitable values of s is known for all values of p (see Refs. 4, 5 and 10) (this is in contrast to the KdV-case, where global-in-time well-posedness is obtained only for $p < 4$ (cf. Ref. 21), and there is finite-time blow-up at least if $p = 4$[23] and strong evidence of blow-up in case $p > 4$[6,7,11]). Results similar to those obtained for KdV-type equations hold in the BBM-context as well. However, stronger results are established in case both the nonlinearity and the dispersion are homogeneous ($G(z) = z^p$ and $\alpha(\xi) = |\xi|^\mu$ where $p = m/n \geq 1$ is rational, m, n relatively prime, n odd and $\mu > 1$). Under these assumptions, a lower bound is derived on the possible decrease in time of the uniform radius of spatial analyticity which is *algebraic* in t. The bound obtained for the KdV-type equations features exponential decrease, as does the bound for the BBM-type equations without the homogeneity assumptions. An intriguing feature of this algebraic bound is that it is asymptotically independent of p as $t \to +\infty$.

2. Loss of Analyticity Detects Loss of Boundedness: The KdV Case

We start with a review of notation which follows that of Ref. 20. For $\tau > 0$, let $A(\tau)$ be the set of all functions f analytic in $S(\tau)$ such that $f \in L_2(S(\tau'))$ for each $0 < \tau' < \tau$, and which are real on the real axis. The collection $A(\tau)$ is a Fréchet space with the $L_2(S(\tau_n))$-norms as the generating system of seminorms, where $0 < \tau_n < \tau$ and $\tau_n \to \tau$ as $n \to \infty$. Thus the topology is defined by the sequence $\{p_k\}_{k=1}^{\infty}$ where $p_k(f) = \int_{-\tau_k}^{\tau_k} \int_{-\infty}^{\infty} |f(x+iy)|^2\, dx\, dy < +\infty$. Standard notation is used for the L_2-based Sobolev classes $H^s = H^s(\mathbb{R})$ and for the L_p-based classes $W_p^k = W_p^k(\mathbb{R})$. An unadorned symbol $\|\cdot\|$ will always denote the usual norm on $L_2 = L_2(\mathbb{R})$. The L_2-inner product will also be written unadorned. Otherwise, the norm in a space X will be denoted $\|\cdot\|_X$. The Fréchet space $H^\infty = H^\infty(\mathbb{R}) = \cap_{s>0} H^s(\mathbb{R})$ will also appear in our analysis. When X denotes one of these Sobolev classes or other topological linear space, and $T > 0$, $C(0, T; X)$ is the class of continuous functions $u : [0, T] \to X$. In case X is a Banach space, $C(0, T; X)$ is equipped with its usual norm

$$\max_{0 \le t \le T} \|u(t)\|_X \,.$$

Interest is first given to the KdV-type equation

$$u_t + G(u)u_x + u_{xxx} = 0 \,. \tag{6}$$

Theorem 1.[20] *Let $T > 0$ and $u \in \cap_{s>0} C(0, T; H^s)$ be a solution of* (6) *where G is real-analytic. If $u_0 \in A(\tau_0)$ for some $\tau_0 > 0$, there is a $\tau_1 > 0$ such that $u \in C(0, T; A(\tau_1))$.*

Hence, starting from analytic initial data, regularity in the L_2-based Sobolev spaces of *all* orders suffices to maintain analyticity. Put another way, if the solution loses analyticity in the sense that it no longer belongs to $A(\sigma)$ for some $\sigma > 0$, then it must have lost its infinite Sobolev regularity. We refer to this situation by saying that loss of analyticity detects loss of H^∞-regularity. In this context, the following result of Albert *et al.*[2] is helpful.

Theorem 2.[2] *Suppose $s \ge 2$ and let $u_0 \in H^s$. Let T^* be the maximum value such that, for all $T \in (0, T^*)$, the solution u of* (6) *with initial data u_0 lies in $C(0, T; H^s)$. Then either $T^* = \infty$ and the solution is global, or*

$$\sup_{0 \le t < T^*} \|u(t)\|_{L_\infty} = \infty \,.$$

Combining Theorems 1 and 2, the following result emerges.

Theorem 3. *Let* $u_0 \in A(\tau_0)$ *for some* $\tau_0 > 0$, *and suppose that the solution* u *of* (6) *with initial data* u_0 *exists on* $[0, T]$ *and satisfies*

$$\sup_{0 \le t \le T} \|u(t)\|_{L_\infty} < \infty \,.$$

Then there exists $\tau_1 > 0$ *such that* $u \in C(0, T; A(\tau_1))$.

Proof. It is easily seen (see Lemma 2.2 in Ref. 20) that $A(\tau) \subset H^\infty$ for any $\tau > 0$, and hence $u_0 \in H^\infty$. Theorem 2 implies that $u \in C(0, T; H^s)$ for all $s > 0$. All the conditions in Theorem 1 are thus satisfied, and it follows that $u \in C(0, T; A(\tau_1))$ for some $\tau_1 > 0$. □

3. Loss of Analyticity Detects Loss of W^1_∞-Regularity: More General Dispersion

Consider (1) where L is homogeneous and defined by $\widehat{Lu}(\xi) = |\xi|^\mu \hat{u}(\xi)$ for some $\mu > 0$, and G is an entire function with power series expansion $G(z) = \sum_{n \ge 0} a_n z^n$ about the origin that is real for real z (so the a_n are all real numbers). Without loss of generality, we may suppose $G(0) = G'(0) = 0$. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(r) = \sum_{n \ge 0} |a_n| r^n$ and remark that the series defining g converges uniformly on bounded subsets and that $|G(z)| \le g(|z|)$ for $z \in \mathbb{C}$. Notice that $\mu = 2$ corresponds to the case treated in Sec. 2. Although Theorem 1 is valid for the more general models, the blow-up criterion presented in Ref. 2 has not been established in such a context. To obtain a result analogous to Therorem 3 in this more general setting, the following proposition will be utilized.

Proposition 4. *Let* u *be a solution of* (1) *defined on some temporal interval* $[0, T]$ *and let* $u_0 = u(\cdot, 0)$ *be its initial value. Assume that there exists* $M > 0$ *such that*

$$\sup_{0 \le t \le T} \|u(\cdot, t)\|_{W^1_\infty} \le M \,. \tag{7}$$

If $u_0 \in H^k$, *for some* $k \in \mathbb{N}$, *then* $u \in C(0, T; H^k)$.

Proof. We argue by induction on k. For clarity, the cases $k = 1, 2, 3$ are treated explicitly and then the inductive step applying for $k \ge 4$ is presented. In the calculations that follow, it is presumed that the solution u and all its partial derivatives lie in L_2. The final inequalities do not depend upon this assumption, but rather feature only Sobolev-norms of u up to order k. In consequence, a standard limiting procedure using local well-posedness allows one to infer the stated inequality under the assumption of limited regularity. Local-in-time well-posedness in H^s for $s > 3/2$, say, follows readily as an application of semigroup theory, as expounded in the work of Kato[19] for example. Also, the constants K appearing below may depend on k, M, T and u_0.

Mutiplying (1) by $u - u_{xx}$, integrating over $\mathbb{R}$, and integrating by parts leads to

$$\frac{1}{2}\frac{d}{dt}\int (u^2 + u_x^2)\,dx = \int F'(u)u_x\,dx - \frac{1}{2}\int G'(u)u_x^3\,dx$$

$$\leq F(u)\Big|_{x=-\infty}^{x=\infty} + \frac{1}{2}g'(M)M\int u_x^2\,dx = K\int u_x^2\,dx\,,$$

where it is presumed $0 \leq t \leq T$ so that (7) holds, $F'(s) = G(s)s$ and here and below, an unadorned integral is taken to be over the entire real axis. Gronwall's lemma yields boundedness of $\|u(\cdot,t)\|_{H^1}$ for $0 \leq t \leq T$.

Multiplying (1) by $u + u_{xxxx}$ and integrating by parts yields

$$\frac{d}{dt}\int (u^2 + u_{xx}^2)\,dx \leq 2\left|\int (G(u)u_x)_{xx}u_{xx}\,dx\right|$$

$$\leq 2\int |G''(u)u_x^2||u_x||u_{xx}|\,dx + 6\int |G'(u)u_x|u_{xx}^2\,dx$$

$$+2\left|\int G(u)\partial_x(u_{xx}^2)\,dx\right|$$

$$\leq K\int (u^2 + u_{xx}^2)\,dx$$

valid at least for $0 \leq t \leq T$. Again, Gronwall's lemma comes to our aid to infer an H^2-bound on this time interval.

Multiplying (1) by $u - u_{xxxxxx}$ and integrating with respect to x for a third time gives, for $0 \leq t \leq T$,

$$\frac{d}{dt}\int (u^2 + u_{xxx}^2)\,dx$$

$$\leq 2\left|\int (G(u)u_x)_{xxx}u_{xxx}\,dx\right|$$

$$\leq 2\int |G'''(u)u_x^3||u_x||u_{xxx}|\,dx + 12\int |G''(u)u_x^2||u_{xx}||u_{xxx}|\,dx$$

$$+7\int |G'(u)u_x|u_{xxx}^2\,dx + 6\left|\int G'(u)u_{xx}^2u_{xxx}\,dx\right|$$

$$\leq K\int (u^2 + u_{xxx}^2)\,dx + 6\left|\int G'(u)u_{xx}^2u_{xxx}\,dx\right|.$$

The last term can be bounded as follows:

$$6\left|\int G'(u)u_{xx}^2u_{xxx}\,dx\right| = 2\left|\int G''(u)u_xu_{xx}^3\,dx\right| \leq K\int |u_{xx}|^3\,dx\,.$$

Agmon's inequality $\|u_{xx}\|_{L_\infty} \le \|u\|_{H^2}^{1/2}\|u\|_{H^3}^{1/2}$ allows us to continue the last inequality as

$$\int |u_{xx}|^3\,dx \le \|u\|_{H^2}^{5/2}\|u\|_{H^3}^{1/2} \le K\|u\|_{H^3}^{1/2}$$

since the boundedness of $\|u\|_{H^2}$ was obtained in the previous step. The boundedness of the H^3-norm of u on $[0,T]$ follows.

The case $k = 3$ shows how the inductive step should be made. Assume the boundedness of u in H^{k-1}. Notice that

$$\|u\|_{L_\infty},\ \|u_x\|_{L_\infty}, \ldots, \|u^{(k-2)}\|_{L_\infty} \le \|u\|_{H^{k-1}}\,, \tag{8}$$

where $u^{(l)}$ is shorthand for $\partial_x^l u$. Multiplying (1) by $u + (-1)^k u^{(2k)}$ leads to

$$\frac{d}{dt}\int (u^2 + (u^{(k)})^2)\,dx \le 2\left|\int (G(u)u_x)^{(k)}u^{(k)}\,dx\right|.$$

Because of (8), the only terms in $(G(u)u_x)^{(k)}$ that can possibly cause a problem are those containing $u^{(k+1)}$, $u^{(k)}$ and $u^{(k-1)}$. These cases are considered separately.
$u^{(k+1)}$*-terms*:

$$\left|\int G(u)u^{(k+1)}u^{(k)}\,dx\right| = \frac{1}{2}\left|\int G(u)\partial_x(u^{(k)})^2\,dx\right| \le K\int (u^{(k)})^2\,dx\,.$$

$u^{(k)}$*-terms*:

$$\left|\int G'(u)u_x u^{(k)}u^{(k)}\,dx\right| \le K\int (u^{(k)})^2\,dx\,.$$

$u^{(k-1)}$*-terms*:

$$\left|\int G''(u)u_x^2 u^{(k-1)}u^{(k)}\,dx\right| \le K\int (u^2 + (u^{(k)})^2)\,dx$$

and

$$\left|\int G'(u)u_{xx}u^{(k-1)}u^{(k)}\,dx\right| \le K\int (u^2 + (u^{(k)})^2)\,dx$$

since $k \ge 4$. □

An analyticity result now follows just as in Sec. 2.

Theorem 5. *Let $u_0 \in A(\tau_0)$ for some $\tau_0 > 0$ and suppose that a solution u of* (1) *satisfies*

$$\sup_{0\le t\le T}\|u(t)\|_{W^1_\infty} < \infty\,.$$

Then there exists $\tau_1 > 0$ such that $u \in C(0,T;A(\tau_1))$.

4. Rate of Decrease of the Uniform Radius of Analyticity for KdV-Type Equations

In this section, it is shown that (1) admits a Gevrey-class analysis. A consequence of this is an explicit lower bound on the possible decrease with time of the uniform radius of analyticity of a solution starting from analytic initial data, and assuming the boundedness of a suitable Sobolev norm.

Let $A = (I - \partial_x^2)^{1/2}$ and, for $\eta \geq 0$, $\tau > 0$, consider the two-parameter family of Gevrey operators $A^\eta e^{\tau A}$. It is easily seen that if $u \in D(A^\eta e^{\tau A})$, the domain of the operator $A^\eta e^{\tau A}$ in L_2, then u is the restriction to the real axis of a function analytic in the complex strip S_τ. In consequence, analyticity results obtain via energy-type estimates in the scale of Gevrey spaces.

Assume the initial-value problem starts with initial data $u_0 \in D(A^\eta e^{\tau_0 A})$ for some positive values of τ_0 and η. Before estimating the resulting solution in a scale of Gevrey spaces, some useful properties of $D(A^\eta e^{\tau A})$ are presented. The first two lemmas are one-dimensional, whole-space versions of the original ones in Ferrari and Titi.[14]

Lemma 6.[14] *Let $\tau \geq 0$ and $\eta > 1/2$. Then $D(A^\eta e^{\tau A})$ is a Banach algebra, which is to say that there is a constant $c = c(\eta)$ such that if $u, v \in D(A^\eta e^{\tau A})$, then*

$$\|A^\eta e^{\tau A}(uv)\| \leq c\|A^\eta e^{\tau A}u\|\|A^\eta e^{\tau A}v\| \,.$$

Remark 7. If $\tau = 0$, this result recovers the standard observation that $H^\eta = D(A^\eta)$ is a Banach algebra for $\eta > 1/2$.

Lemma 8.[14] *Let $\tau \geq 0$ and $\eta > 1/2$ be given. Suppose F is an entire function with Taylor series $\sum_{n=0}^\infty a_n z^n$ at 0. If $u \in D(A^\eta e^{\tau A})$, then there are positive constants $c_1 = c_1(\eta)$ and $c_2 = c_2(\eta)$ such that*

$$\|A^\eta e^{\tau A} F(u)\| \leq c_1 f(c_2\|A^\eta e^{\tau A}u\|) \,,$$

where $f(r) = \sum_{n=0}^\infty |a_n| r^n$.

The next lemma is an interpolation inequality that may be found for example in Ref. 24.

Lemma 9. *Let r, s and τ be non-negative numbers. Then there are absolute constants c_1 and c_2 such that for $z \in D(A^{r+s} e^{\tau A})$,*

$$\|A^r e^{\tau A} z\| \leq c_1\|A^r z\| + c_2 \tau^s \|A^{r+s} e^{\tau A} z\| \,.$$

The following lemma exploits some cancellation properties inherent in the nonlinear term. A multidimensional version in $D(A^r)$ was first presented in Ref. 12 (in a study of the Navier–Stokes nonlinearity), and a generalization to $D(A^r e^{\tau A})$ was given in Ref. 22 where analyticity properties of a generalized Euler equation were studied. This is a version of Lemma 8 in Ref. 22 suitable for our purposes.

Lemma 10. *Let $r > 3/2$ and $\tau \geq 0$. For $w \in D(A^{r+1}e^{\tau A})$ and $z \in D(A^{r+1/2}e^{\tau A})$, there are constants c_1 and c_2 depending only on r such that*

$$|(A^r e^{\tau A}(w\partial_x z), A^r e^{\tau A} z)| \leq c_1\|A^{r+1}w\|\|A^r z\|^2 + c_2\tau\|A^{r+1}e^{\tau A}w\|\|A^{r+1/2}e^{\tau A}z\|^2 .$$

Proof. Define I_1 and I_2 by the formula

$$\begin{aligned}
&(A^r e^{\tau A}(w\partial_x z), A^r e^{\tau A} z) \\
&\quad = [(A^r e^{\tau A}(w\partial_x z), A^r e^{\tau A} z) - (w\partial_x(A^r e^{\tau A} z), A^r e^{\tau A} z)] \\
&\quad\quad + (w\partial_x(A^r e^{\tau A} z), A^r e^{\tau A} z) = I_1 + I_2 .
\end{aligned} \tag{9}$$

Integrating by parts, one obtains that there is a constant c depending only on $r > 1/2$ for which

$$\begin{aligned}
I_2 &= -\frac{1}{2}((A^r e^{\tau A} z)^2, \partial_x w) \leq \frac{1}{2}\|\partial_x w\|_{L^\infty}\|A^r e^{\tau A} z\|^2 \\
&\leq c\|A^r\partial_x w\|\|A^r e^{\tau A} z\|^2 \leq c\|A^{r+1}w\|\|A^r e^{\tau A} z\|^2 .
\end{aligned} \tag{10}$$

By Lemma 9, there are constants c_1 and c_2 such that

$$\|A^r e^{\tau A} z\|^2 \leq c_1\|A^r z\|^2 + c_2\tau\|A^{r+1/2}e^{\tau A}z\|^2 .$$

Inserting the last inequality in (10) gives the desired bound on I_2.

A bound on I_1 presented in Ref. 22 is valid in any space dimension and could be incorporated directly. However, guided by the application in view, we modify the last two estimates in Ref. 22 trading a smaller power for a less restrictive assumption on r. More precisely, we replace the estimate (43) in Lemma 8 of Ref. 22 with the inequality

$$\begin{aligned}
&\int |\xi|^{3/2} e^{\tau(1+|\xi|^2)^{1/2}} |\hat{z}(\xi)|\, d\xi \\
&\quad \leq \int ([1+|\xi|^2]^{1/2})^{3/2} e^{\tau(1+|\xi|^2)^{1/2}} |\hat{z}(\xi)|\, d\xi \\
&\quad \leq \left(\int ([1+|\xi|^2]^{1/2})^{2-2r}\, d\xi\right)^{1/2} \\
&\quad\quad \times \left(\int ([1+|\xi|^2]^{1/2})^{2r+1} e^{2\tau(1+|\xi|^2)^{1/2}} |\hat{z}(\xi)|^2\, d\xi\right)^{1/2} \\
&\quad = c\|A^{r+1/2}e^{\tau A}z\| .
\end{aligned} \tag{11}$$

Of course the assumption $r > 3/2$ is crucial to this estimate. The stated inequality is thereby derived for I_1 and the lemma established. □

The stage is now set for $D(A^\eta e^{\tau A})$-estimates. It will be convenient to consider the equation satisfied by $v = u_x$ obtained by differentiating (1), namely

$$v_t + G'(u)v^2 + G(u)v_x - (Lv)_x = 0\,. \tag{12}$$

Let τ be a decreasing, positive function of time that is at least C^1 and such that $\tau(0) = \tau_0$. Taking the L_2-inner product of $A^{2\eta}e^{2\tau A}v$ with Eq. (12) yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|A^\eta e^{\tau A}v\|^2 &- \dot{\tau}\|A^{\eta+1/2}e^{\tau A}v\|^2 \\ &\le |(A^\eta e^{\tau A}(G'(u)v^2), A^\eta e^{\tau A}v)| + |(A^\eta e^{\tau A}(G(u)v_x), A^\eta e^{\tau A}v)| \\ &= NL_1 + NL_2\,.\end{aligned} \tag{13}$$

First, the term NL_1 is bounded in the following way:

$$\begin{aligned}NL_1 &\le \|A^\eta e^{\tau A}(G'(u)v^2)\|\|A^\eta e^{\tau A}v\| \le c\|A^\eta e^{\tau A}G'(u)\|\|A^\eta e^{\tau A}v\|^3 \\ &\le c_1 g'(c_2\|A^\eta e^{\tau A}u\|)\|A^\eta e^{\tau A}v\|^3 \le c_1\sum_{n=1}^{\infty} n|a_n|c_2^{n-1}\|A^\eta e^{\tau A}v\|^{n+2}\,,\end{aligned} \tag{14}$$

where c_1 and c_2 are constants depending only on η as indicated by Lemmas 6 and 8 which were used in the second and the third step, respectively. Applying Lemma 9 with $z = v$, $r = \eta$, $s = 1/(n+2)$ and raising the resulting inequality to the $(n+2)$th-power leads to

$$\|A^\eta e^{\tau A}v\|^{n+2} \le 2^{n+1}c_1^{n+1}\|A^\eta v\|^{n+2} + 2^{n+1}c_2^{n+1}\tau\|A^{\eta+1/(n+2)}e^{\tau A}v\|^{n+2}\,. \tag{15}$$

Interpolating the quantity on the right-hand side of (15), viz.

$$\|A^{\eta+1/(n+2)}e^{\tau A}v\| \le \|A^\eta e^{\tau A}v\|^{n/(n+2)}\|A^{\eta+1/2}e^{\tau A}v\|^{2/(n+2)}\,, \tag{16}$$

and inserting this into (15) gives, via (14), the inequality

$$\begin{aligned}NL_1 &\le c_1\left(\sum_{n=1}^{\infty} n|a_n|c_2^{n-1}\|A^\eta v\|^{n-1}\right)\|A^\eta v\|^3 \\ &\quad + c_3\tau\left(\sum_{n=1}^{\infty} n|a_n|c_4^{n-1}\|A^\eta e^{\tau A}v\|^{n-1}\right)\|A^\eta e^{\tau A}v\|\|A^{\eta+1/2}e^{\tau A}v\|^2 \\ &= c_1 g'(c_2\|A^\eta v\|)\|A^\eta v\|^3 + c_3\tau g'(c_4\|A^\eta e^{\tau A}v\|)\|A^\eta e^{\tau A}v\|\|A^{\eta+1/2}e^{\tau A}v\|^2\,,\end{aligned} \tag{17}$$

where c_1, c_2, c_3 and c_4 are constants depending only on η.

Since NL_2 contains v_x, it cannot be handled in the same manner. However, a bound comparable to (17) is not unexpected assuming more regularity (a larger value of η). Notice that NL_2 has the form

$$(A^r e^{\tau A}(w\partial_x z), A^r e^{\tau A}z)\,. \tag{18}$$

Assuming $\eta > 3/2$, Lemma 10 may be applied with $w = G(u)$, $z = v$, $r = \eta$. There follow the inequalities

$$\begin{aligned} NL_2 &\le c_1\|A^{\eta+1}G(u)\|\|A^\eta v\|^2 + c_2\tau\|A^{\eta+1}e^{\tau A}G(u)\|\|A^{\eta+1/2}e^{\tau A}v\|^2 \\ &\le c_1 g(c_2\|A^{\eta+1}u\|)\|A^\eta v\|^2 + c_3\tau g(c_4\|A^{\eta+1}e^{\tau A}u\|)\|A^{\eta+1/2}e^{\tau A}v\|^2 \\ &\le c_1 g(c_2\|A^\eta v\|)\|A^\eta v\|^2 + c_3\tau g(c_4\|A^\eta e^{\tau A}v\|)\|A^{\eta+1/2}e^{\tau A}v\|^2\,, \end{aligned} \tag{19}$$

where c_1, c_2, c_3 and c_4 are constants depending only on η.

Collecting the estimates (17) and (19), one obtains

$$\begin{aligned} &\frac{d}{dt}\|A^\eta e^{\tau A}v\|^2 + (-\dot\tau - c_1\tau h(\|A^\eta e^{\tau A}v\|))\|A^{\eta+1/2}e^{\tau A}v\|^2 \\ &\qquad \le c_2 h(\|A^\eta v\|)\|A^\eta v\|^2 \end{aligned} \tag{20}$$

with $h(s) = g(c_3 s) + s g'(c_4 s)$ and c_1, c_2, c_3 and c_4 constants depending only on η. From the differential inequality (20) it is straightforward to advance the conclusion $v(\cdot,t) \in D(A^\eta e^{\tau(t)A})$ for $0 \le t \le T$ where

$$\begin{aligned} \tau(t) = \tau_0 \exp\Bigg[- c_1 \int_0^t h\Bigg(&\|A^\eta e^{\tau_0 A}v_0\|^2 \\ &+ c_2 \int_0^{t'} h(\|A^\eta v(t'')\|)\|A^\eta v(t'')\|^2\,dt''\Bigg)\,dt'\Bigg]\,. \end{aligned} \tag{21}$$

Expressing this in terms of u yields the final result. □

Theorem 11. *Let G be an entire function with Taylor series expansion $\sum_{n=0}^\infty a_n z^n$ about 0 and let $g(r) = \sum_{n=0}^\infty |a_n| r^n$ for $r \ge 0$. Let $A = (I - \partial_x^2)^{1/2}$ and suppose $u_0 \in D(A^\eta e^{\tau_0 A})$ for some $\tau_0 > 0$ and $\eta > 5/2$. Assume that the solution u of (1) lies in $C(0,T;D(A^\eta))$ for some $T > 0$. Then $u \in C(0,T;D(A^\eta e^{\tau(t)A}))$ where $\tau(t)$ is as depicted in (21) with $h(s) = g(c_3 s) + s g'(c_4 s)$ and c_1, c_2, c_3 and c_4 constants depending only on η.*

Remark 12. If $G(z) = z^p$ is a pure power, then $h(s) = c_0 s^p$ for some constant c_0. Consequently, τ has the form

$$\tau(t) = \tau_0 \exp(-\gamma(t))\,,$$

where

$$\gamma(t) = \int_0^t c_0 \left[d + c_2 c_0 \int_0^{t'} \|A^\eta u(\cdot,t'')\|^{p+2}\,dt'' \right]^p dt'$$

and $d = \|A^\eta e^{\tau_0 A}u_0\|$.

Remark 13. In the case of the generalized KdV-equation ($L = -\partial_x^2$ and $G(z) = z^p$), no *a priori* Sobolev boundedness is needed if $p < 4$. Namely, if $u_0 \in D(A^\eta e^{\tau_0 A})$

(η, τ_0 as in the theorem), then $u_0 \in D(A^\eta)$ and since the generalized KdV-equation is globally well-posed for this choice of the parameters (cf. Ref. 21), $u \in C(0,T;D(A^\eta))$ for all $T > 0$. Hence, for $p < 4$ in the generalized KdV-equation, a solution that starts analytic in a strip remains so for all time. However, for $p \geq 4$, *a priori* Sobolev bounds are not available for large initial data. Indeed, for $p = 4$ we know that solutions blow up in finite time (see e.g. Refs. 3 and 23). Moreover, for $p > 4$, numerically obtained evidence and some theory suggest that blow-up occurs for all $p \geq 4$.[6,7,11] In these contexts, our theory shows a solution initiated with a disturbance that is analytic in a strip remains analytic in a, possibly smaller, strip up to the time of blow-up. Obviously, the strip of analyticity must shrink to zero as t approaches the blow-up time.

5. Rate of Decrease of the Uniform Radius of Analyticity for BBM-Type Equations

Consider (2) with the polynomial nonlinearity $G(z) = z + z^p$ and the homogeneous dispersion, $\alpha(\xi) = |\xi|^\mu$ where $p = m/n \geq 1$ is rational, m, n relatively prime and n odd and $\mu > 1$, and rewrite it in the form

$$(I + L)u_t + \left(u + \frac{1}{p+1}u^{p+1}\right)_x = 0\,. \tag{22}$$

The point of this assumption on p is that, by a suitable choice of branch of the map $z \to z^{1/n}$, it can be assumed that G is real on the real axis. Recall that the symbol of L is $|\xi|^\mu$, and so the operator $I + L$ can be written as A^μ where the symbol of A is $(1 + |\xi|^\mu)^{1/\mu}$. Note that in terms of regularity, A is equivalent to $(I - \partial_x^2)^{1/2}$.

The initial data u_0 will be taken from a Geverey class, viz.

$$u_0 \in D(A^{\varepsilon + \mu/2} e^{\tau_0 A}) \tag{23}$$

for some $\varepsilon, \tau_0 > 0$. A suitable value for ε will be determined presently. Let u be a solution of (22) with initial data u_0. As $\mu > 1$, u is globally defined and lies at least in $C(0,T;H^{\mu/2+\varepsilon})$ for any $T > 0$.

As in the last section, let $\tau(t)$ be a positive, decreasing, C^1-function such that $\tau(0) = \tau_0$. Our goal is to obtain *a priori* estimates on the Gevrey-class norms $\|A^{\varepsilon+\mu/2}e^{\tau A}\cdot\|$ as we did for KdV-type equations in Sec. 4. Taking the L_2-inner product of (22) with $A^{2\varepsilon}e^{2\tau A}u$ followed by some straightforward manipulations, there appears the differential inequality

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|A^{\varepsilon+\mu/2}e^{\tau A}u\|^2 - \dot{\tau}\|A^{\varepsilon+\mu/2+1/2}e^{\tau A}u\|^2 \\ &\qquad \leq c\|A^{\varepsilon+1/2}e^{\tau A}u^{p+1}\|\|A^{\varepsilon+1/2}e^{\tau A}u\|\,. \end{aligned} \tag{24}$$

Since the spaces $D(A^\eta e^{\tau A})$ are Banach algebras for $\eta > 1/2$, the right-hand side of (24) may be bounded by $c\|A^{\varepsilon+1/2}e^{\tau A}u\|^{p+2}$ for a suitable constant c possibly depending on p, μ and ε. Taking that into account and exploiting the interpolation inequality presented in Lemma 9, (24) leads to

$$\frac{1}{2}\frac{d}{dt}\|A^{\varepsilon+\mu/2}e^{\tau A}u\|^2 - \dot{\tau}\|A^{\varepsilon+\mu/2+1/2}e^{\tau A}u\|^2$$
$$\le c\|A^{\varepsilon+1/2}u\|^{p+2} + c\tau^{s(p+2)}\|A^{\varepsilon+1/2+s}e^{\tau A}u\|^{p+2}\,. \tag{25}$$

Next, interpolate the last term in (25) as follows. For any s such that

$$\mu/2 \le 1/2 + s \le \mu/2 + 1/2\,,$$

write

$$\|A^{\varepsilon+1/2+s}e^{\tau A}u\| \le \|A^{\varepsilon+\mu/2}e^{\tau A}u\|^{-2s+\mu}\|A^{\varepsilon+\mu/2+1/2}e^{\tau A}u\|^{1+2s-\mu}\,.$$

Using this in (25) yields

$$\frac{1}{2}\frac{d}{dt}\|A^{\varepsilon+\mu/2}e^{\tau A}u\|^2 - \dot{\tau}\|A^{\varepsilon+\mu/2+1/2}e^{\tau A}u\|^2 \le c\|A^{\varepsilon+1/2}u\|^{p+2}$$
$$+ c\tau^{s(p+2)}\|A^{\varepsilon+\mu/2}e^{\tau A}u\|^{(-2s+\mu)(p+2)}\|A^{\varepsilon+\mu/2+1/2}e^{\tau A}u\|^{(1+2s-\mu)(p+2)}\,. \tag{26}$$

The specification

$$s = \frac{\mu-1}{2} + \frac{1}{p+2}$$

optimizes the exponents, and with this choice the differential inequality (26) becomes

$$\frac{1}{2}\frac{d}{dt}\|A^{\varepsilon+\mu/2}e^{\tau A}u\|^2 + c(-\dot{\tau} - \tau^{1+\frac{(\mu-1)(p+2)}{2}}\|A^{\varepsilon+\mu/2}e^{\tau A}u\|^p)\|A^{\varepsilon+\mu/2+1/2}e^{\tau A}u\|^2$$
$$\le c\|A^{\varepsilon+1/2}u\|^{p+2}\,. \tag{27}$$

Recall that $\varepsilon > 0$ was arbitrary in the preceding ruminations. Since $\mu > 1$, we can set $\varepsilon = \mu/2 - 1/2$. With this value of ε, the right-hand side of (27) becomes $c\|A^{\mu/2}u\|^{p+2}$. The advantage of this choice of ε is that the L_2-norm of $A^{\mu/2}u$ is a conserved quantity as is readily seen by multiplying (22) by u. Hence, the right-hand side of the differential inequality (27) is equal to a quantity independent of time, namely to $c\|A^{\mu/2}u_0\|^{p+2}$. The inequality (27) can be analyzed in the following manner. As long as

$$-\dot{\tau} - \tau^{1+\frac{(\mu-1)(p+2)}{2}}\|A^{\varepsilon+\mu/2}e^{\tau A}u\|^p \ge 0\,, \tag{28}$$

(27) implies

$$\frac{d}{dt}\|A^{\varepsilon+\mu/2}e^{\tau A}u\|^2 \le c\|A^{\varepsilon+1/2}u_0\|^{p+2}\,,$$

and thus

$$\|A^{\varepsilon+\mu/2}e^{\tau A}u\| \le \|A^{\varepsilon+\mu/2}e^{\tau_0 A}u_0\| + c\|A^{\varepsilon+1/2}u_0\|^{(p+2)/2}t^{1/2}\,. \tag{29}$$

Inserting this into (28), it is concluded that (29) will certainly hold as long as

$$\dot{\tau} \le -[\|A^{\varepsilon+\mu/2}e^{\tau_0 A}u_0\| + c\|A^{\varepsilon+1/2}u_0\|^{(p+2)/2}t^{1/2}]^p \tau^{1+\frac{(\mu-1)(p+2)}{2}}\,. \tag{30}$$

Straightforward integration yields the following lower bound for τ; for all $t \ge 0$,

$$\tau(t) \ge \frac{1}{K}\frac{1}{\tau_0^{-1} + \sum_{k=0}^{k=p} t^{\frac{p-k+2}{(\mu-1)(p+2)}}}\,, \tag{31}$$

where K is a constant depending on μ, p, τ_0 and u_0. Consequently, asymptotically in time, τ is bounded below by a positive constant times

$$t^{-\frac{1}{\mu-1}}\,. \tag{32}$$

A very interesting feature of (32) is its *independence* of the strength of the nonlinearity p. Hence, the influence of the degree of the nonlinearity weakens as t gets larger, and vanishes in the limit as $t \to \infty$.

The results obtained in this section are summarized in the following theorem.

Theorem 14. *Consider* (2) *with the polynomial nonlinearity* $G(z) = z + z^p$ *and homogeneous dispersion* $\alpha(\xi) = |\xi|^\mu$ *where* $\mu > 1$. *Assume that for some* $\tau_0 > 0$ *the initial data* u_0 *belongs to the Gevrey class* $D(A^{\mu-1/2}e^{\tau_0 A})$. *Then, the solution* u *of* (2) *with initial value* u_0 *remains analytic for all positive times. More precisely,* $u \in C(0,T; D(A^{\mu-1/2}e^{\tau A}))$ *for all* $T > 0$ *where a lower bound on the uniform radius of analyticity* $\tau(t)$ *is given by* (31). *Moreover, asymptotically in time* (*as* $t \to \infty$), τ *is bounded below by an algebraic expression of the form* $t^{-\frac{1}{\mu-1}}$ *which is independent of* p.

Remark 15. A lower bound on the rate of decrease of the uniform radius of analyticity for the more general BBM-type models of the form (2) features exponential decrease — the proof is completely analogous to the proof presented in Sec. 4 for the general KdV-type models.

Remark 16. A natural question to ask is whether it is possible to derive an algebraic lower bound for the KdV-type models at least in the presence of homogeneous nonlinearity and dispersion provided the dispersive effect is strong enough (in particular, for the generalized KdV equation itself). One indication that the answer could be affirmative is the algebraic bounds for the growth of norms of derivatives in time for the generalized KdV equation obtained in Ref. 25. The setting of Bourgain–Gevrey spaces (see Ref. 17) may be suitable here since Bourgain–Gevrey spaces seem to be well-designed for exploiting dispersive effects in Gevrey classes. This will be addressed in a future work.

Acknowledgments

This research was partially supported by the US National Science Foundation. The manuscript was completed while the first author was visiting the Institute for Mathematical Sciences at the Chinese University of Hong Kong.

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