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**Journal of Mathematical Fluid
Mechanics**

ISSN 1422-6928

Volume 16

Number 2

J. Math. Fluid Mech. (2014) 16:321-334

DOI 10.1007/s00021-013-0155-0



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Blow-Up Scenarios for the 3D Navier–Stokes Equations Exhibiting Sub-Criticality with Respect to the Scaling of One-Dimensional Local Sparseness

Zachary Bradshaw and Zoran Grujić

Communicated by G.P. Galdi

Abstract. It is shown that, if the vorticity magnitude associated with a (presumed singular) three-dimensional incompressible Navier–Stokes flow blows-up in a manner exhibiting certain *time dependent local structure*, then *time independent* bounds on the L^1 norm of $|\omega| \log \sqrt{1 + |\omega|^2}$ follow. The implication is that the volume of the region of high vorticity decays at a rate of greater order than a rate connected to the critical scaling of one-dimensional local sparseness and, consequently, the solution becomes sub-critical.

Mathematics Subject Classification (2010). Primary 35B65; Secondary 35B44.

Keywords. Navier–Stokes equations, regularity, criticality, blow-up rates.

1. Introduction

The purpose of this article is to present blow-up scenarios under which Leray solutions to the 3D Navier–Stokes equations (3D NSE) behave *sub-critically* with respect to the critical scaling of *one-dimensional local sparseness*. In order to contextualize our discussion we begin with several general remarks. Speaking informally, there is a scaling-gap between known *a priori* finite quantities on one hand, e.g. $\sup_{0 < t < T} \|u\|_2^2$ and $\sup_{0 < t < T} \|\nabla u\|_1$, and, on the other hand, quantities with respect to which regularity can be conditioned, e.g. $\sup_{0 < t < T} \|u\|_3^3$ and $\sup_{0 < t < T} \|\nabla u\|_2^2$. Under the natural scaling for 3D NSE,

$$u(x, t) \mapsto \frac{1}{\lambda} u_\lambda \left(\frac{x}{\lambda}, \frac{t}{\lambda^2} \right),$$

we see that the *a priori* controlled quantities exhibit sub-critical scaling—our examples scale as λ^1 —while those sufficient for regularity scale critically as λ^0 . This mismatch is referred to as the “scaling-gap” and indicates the Navier–Stokes problem is super-critical.

A theme apparent in many regularity results is the inclusion of premises that *explicitly bridge* the scaling gap. For example, a significant result in regularity theory was non-existence of backward-in-time self-similar blow-up (cf. [15, 19, 23]). Because self-similar solutions are scaling invariant in virtue of their construction, their study is effectively a restriction to a class of critical solutions. Indeed, self-similar solutions satisfy a scaling invariant point-wise bound, $\text{ess sup} \left((|x - x_0| + \sqrt{T - t}) |u(x, t)| \right) < \infty$, where the essential supremum is taken over an appropriate parabolic cylinder. For non-self-similar solutions satisfying this critical estimate, regularity has been affirmed in the axisymmetric case (cf. [3, 4, 17, 20]; the general case remains open). Here, the general theme is apparent as, first, the flow is assumed to exhibit some feature which is critical and, second, additional restrictions are identified from which regularity follows. Similar examples, in particular those from the ϵ -regularity theory, require additional conditions on the smallness of the presumed finite scale-invariant quantity.

In two complementary publications, [8, 13], a new *dynamic* approach to bridging the scaling gap is elucidated. The physical motivation is the persistence (in the average sense) of the axial lengths of vortex filaments. This picture is supported in experimental and numerical studies and by a mathematical result which is the main result of [8]. By considering this length persistence and the *decay rate for the volume*

of the region of intense vorticity, a connection is found between the scaling of the latter quantity and the critical scaling of one-dimensional local sparseness. As this argument provides the context for our own results we make the matter more precise.

The rigorous regularity criteria is presented in [13]. In that paper, regularity of mild solutions to 3D NSE with $L^\infty(\mathbb{R}^3)$ initial data is conditioned on a geometric measure-type criteria involving one-dimensional linear sparseness of the super-level sets of the vorticity. The implication is that the critical scaling for the local one-dimensional sparseness of the region of intense vorticity is of order $c_0 \|\omega(t)\|_\infty^{-1/2}$. The proof of this geometric measure-type regularity criteria is based on an interplay among the diffusion, the basic symmetries present in the 3D NSE, and geometric properties of the harmonic measure (resulting in strong anisotropic diffusion); this in turn draws on the ideas exposed in [12]. On the other hand, the a priori estimates on the L^1 norm of the vorticity found in [6, 18] imply the volume of the region where vorticity magnitudes are high decays according to,

$$\text{Vol} \left(\Lambda_t \left(\frac{1}{c_1} \|\omega(t)\|_\infty \right) \right) \leq \frac{c_2}{\|\omega(t)\|_\infty},$$

where $\Lambda_t(y) = \{x : |\omega(x, t)| \geq y\}$. Interestingly, if the region of intense vorticity corresponds precisely to the space occupied by filamentary vortex structures and the length of these filaments is non-decreasing, e.g. if they are pinned to the characteristic length scale of a turbulent region, then the anti-axial diameters of these filaments exhibit a rate of decay of order at least $c_3 \|\omega(t)\|_\infty^{-1/2}$, which matches the critical scaling for local one-dimensional sparseness. It is in this sense that the problem is rendered critical. That the vortex filaments have persistent lengths (in the average) which are comparable to the scale of the turbulent region is not yet rigorously established but is supported by numerical evidence as well as a mathematical evidence which is the prime consideration of [8].

Even if this argument is sound, it is not immediately evident that regularity follows (it would if one could reconcile the possible difference between c_0 and c_3). The present paper partially overcomes this by illustrating two blow-up scenarios in which the region of intense vorticity decays at a faster rate than the critical rate discussed in [8, 13], thereby rendering the constants irrelevant. In particular, we are interested in concluding,

$$\text{Vol} \left(\Lambda_t \left(\frac{1}{c_1} \|\omega(t)\|_\infty \right) \right) \leq \frac{c_3}{\|\omega(t)\|_\infty \Phi(\|\omega(t)\|_\infty)},$$

where, in our first scenario, $\Phi(x) = \log(1+x)$, and, in the second, $\Phi(x) = \log \log(e+x)$. These results are given in Sect. 3. We note for clarity that ours are not stated as regularity criteria but instead scenarios under which 3D NSE becomes sub-critical with regard to the scaling described in [8, 13].

The above decay rates will be obtained by imposing certain structural requirements on the blow-up rates exhibited by vorticity components. Let ω_j^+ and ω_j^- respectively denote the positive and negative vorticity components truncated away from zero. We will define amenable blow-up rates of orders 0 and 1 in detail in Definition 6 and here only illustrate a class of functions exhibiting such blow-up profiles and describe how the blow-up rates are connected to ω_j^\pm . A function g (to be identified with one of the ω_j^\pm s) exhibits a local algebraic blow-up (around x_0 , at time T) if there exists a constant $C > 1$ so that, for (x, t) in a parabolic cylinder, $Q = B(x_0, r) \times (0, T)$, we have,

$$\frac{1}{C} \left(\frac{1}{|p(x, t)| + \tau(t)} \right)^{\alpha(t)} \leq |g(x, t)| \leq C \left(\frac{1}{|p(x, t)| + \tau(t)} \right)^{\alpha(t)},$$

where, at each time t , $p(\cdot, t)$ is some polynomial of degree less than a fixed natural number d , α is positive valued and bounded away from both 0 and ∞ , and τ is a positive (up to T) scalar function of time which vanishes at the singular time. The envisioned blow-up occurs at the zeros of $p(x, T)$ lying within $B(x_0, r)$. There is a considerable amount of freedom present in the above construction as the polynomial is allowed to vary wildly in the time dimension. We will also consider a scenario where some asymmetry is allowed between the bounds assumed on vorticity components. In particular, for appropriate blow-up

rates $D_j^\pm(x, t)$, we will require,

$$D_j^\pm(x, t) \leq |\omega_j^\pm(x, t)| \leq CD_j^\pm(x, t)^{\beta_j^\pm(t)},$$

where $\beta_j^\pm : (0, T) \rightarrow [1, B_j]$ for some fixed value B_j .

A key role is played in our analysis by the cancellations evident in the vortex stretching term in the context of the real Hardy space \mathcal{H}^1 exploited via the Div-Curl lemma [5] and the $\mathcal{H}^1 - BMO$ duality [10, 11]. In the standard way this gives uniform-in-time control of the vortex stretching term. The structural blow-up assumptions are provided to ensure uniform-in-time control of the BMO norm of a multiplier (multiplied against the vortex stretching term). In the algebraic case, this is enabled by the logarithm's depletive effect on the unboundedness of the mean oscillations of polynomial functions. The effectiveness is witnessed by the remarkable fact (cf. a proof by Stein in [21]) that there exists a constant $C(d, n)$ so that, for any polynomial on \mathbb{R}^n of degree less than or equal to d ,

$$\|\log |P|\|_{BMO} \leq C(d, n).$$

In particular, the constant is *independent of the coefficients*. This allows us to introduce *time-dependent algebraic comparability conditions* on the spatial profiles prior to a possibly singular time and do so in a manner that preserves *time-independent estimates* on the BMO norms of the logarithms of these profiles.

We proceed in Sect. 2 to review needed results from harmonic analysis and then define the classes of blow-up scenarios which will be amenable to our PDE argument. We also include results which connect the structural blow-up assumptions to an energy inequality-type argument given in Sect. 3. The statements and proofs of the main results are contained in Sect. 3.

2. Preliminaries and Amenable Blow-Up Rates

Here we review needed results from harmonic analysis and present a lemma which will connect these ideas to the PDE context of Sect. 3. Following [22], the maximal function of a distribution f is defined for all $x \in \mathbb{R}^n$ as,

$$M_h f(x) = \sup_{t>0} |f * h_t(x)|,$$

where h is a fixed test function supported on the unit ball so that $\int h \, dx = 1$ and h_t denotes $t^{-n}h(x/t)$.

Definition 1. The distribution f is in the Hardy space \mathcal{H}^1 if $\|f\|_{\mathcal{H}^1} := \|M_h f\|_1 < \infty$.

In [5], Coifman, Lions, Meyer, and Semmes reformulated and refined some key features of the ‘sequential’ theory of compensated compactness within the framework of Hardy spaces, the key idea being that certain *nonlinear quantities* exhibit *cancellations* which lead to greater regularity. One such result is the Div-Curl lemma.

Lemma 2. (Coifman, Lions, Meyer, Semmes—[5]) *Suppose $E, B \in L^2(\mathbb{R}^3)^3$ with $\nabla \cdot E = 0$ and $\nabla \times B = 0$ (in the sense of distributions). Then, $E \cdot B \in \mathcal{H}^1$, and there exists a universal constant C such that*

$$\|E \cdot B\|_{\mathcal{H}^1} \leq C\|E\|_{L^2}\|B\|_{L^2}.$$

Because weak solutions to the Navier–Stokes equations are divergence free and because the curl of a gradient is always equal to the trivial distribution, it follows that the advective term, $(u \cdot \nabla)u$, in the velocity-pressure formulation exhibits div-curl structure, as does the vortex stretching term, $(\omega \cdot \nabla)u$, in the vorticity-velocity formulation. Consequently, using Hardy spaces, refined regularity results can be established for weak solutions of 3D NSE (see Chapter 3.2 of [18] for a collection of such results).

The well known result of Fefferman (cf. [10, 11] as well as Stein’s monograph [22]) establishes that the dual space of \mathcal{H}^1 is precisely the space of functions of bounded mean oscillation, BMO . By f_B we denote the average of the locally integrable function f over the ball B of volume $|B|$; i.e., $f_B = |B|^{-1} \int_B f \, dx$. The mean oscillation of f over B is $|B|^{-1} \int |f - f_B| \, dx$ and is the quantity typically used to characterize BMO .

Definition 3. The locally integrable function f is in BMO if $\|f\|_{BMO} < \infty$ where

$$\|f\|_{BMO} := \sup_{x \in \mathbb{R}^3; 0 < r} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y) - f_{B(x, r)}| \, dy.$$

Several results about the space BMO will expedite future work. An elementary fact is that, if $g \in BMO$, then the following implication holds,

$$g(x) + m \leq f(x) \leq g(x) + M \text{ for a. e. } x \in \mathbb{R}^3 \implies \|f\|_{BMO} \leq \|g\|_{BMO} + M - m. \tag{1}$$

The above is easy to see by directly comparing $|f - f_B|$ to $|g + M - (g + m)_B|$ and concluding that, for all balls B , we have,

$$\frac{1}{|B|} \int_B |f - f_B| \, dx \leq \frac{1}{|B|} \int_B |g - g_B| \, dx + M - m.$$

We include a technical lemma regarding certain point-wise multipliers on BMO which appears in a paper by Iwaniec and Verde (cf. Lemma 2.1 of [16]; see also Lemma 5.10 of [2]). This result will allow us to use the standard \mathcal{H}^1 - BMO duality in an essentially local context instead of working in the local versions of these spaces. It is worth mentioning that our results can be achieved in the local spaces using a local non-homogeneous version of the Div-Curl lemma (cf. [5]) paired with various local $h^1 - bmo$ dualities (cf. [2]) thereby obtaining estimates on the localized vortex stretching term (this approach is similar to the estimates in [14]).

Lemma 4. (Iwaniec and Verde—[16]) *Suppose $h \in BMO(\mathbb{R}^n)$ and $\phi \in C_0^1(\mathbb{R}^n)$ is supported on the ball B . Then $\phi h \in BMO(\mathbb{R}^n)$ and we have,*

$$\|\phi h\|_{BMO} \leq C(n, R) \|\nabla \phi\|_\infty \left(\|h\|_{BMO} + \left| \frac{1}{|B|} \int_B h \, dx \right| \right).$$

The space BMO enjoys an intimate connection with the logarithm. This is illustrated in the following lemma due to Stein.

Lemma 5. (Stein—[21]) *Let P be any polynomial in \mathbb{R}^n of degree less than or equal to d . Then there exists a constant $C = C(d, n)$ so that $\log |P| \in BMO$ and*

$$\|\log |P|\|_{BMO} \leq C(d, n).$$

The fact that the constant appearing above is *independent of the coefficients* evidences the remarkable extent to which the logarithm depletes mean oscillations and motivates the assumptions on which the conclusions of Sect. 3 are conditioned. This will be done by assuming that solutions blow-up in a fashion possessing certain structure. The remainder of this section is dedicated to describing amenable blow-up structures and the statement and proof of a lemma which will connect this structure to the PDE context of 3. We denote by $\log^m(x)$ an iterated composition of the logarithm with itself $m - 1$ times, that is, $\log^m(x) = \log \circ \dots \circ \log(\sqrt{e_m + x^2})$ where m applications the natural logarithm are carried out and e_m is a power of e defined so that $\log^m(0) = 0$. We will use the following notation when defining functions of (x, t) possessing singularities at time T in the spatial set $S \subset \mathbb{R}^3$,

$$Q_T(x_0, R, S) = B(x_0, R) \times (0, T) \cup ((B(x_0, R) \setminus S) \times \{T\}),$$

and,

$$\Omega_T(S) = \mathbb{R}^3 \times (0, T) \cup (\mathbb{R}^3 \setminus S) \times \{T\},$$

where $R > 0$ and $x_0 \in \mathbb{R}^3$. In contexts where x_0 and R are fixed we make the abbreviation $Q_T(S) = Q_T(x_0, R, S)$.

Definition 6. Fix $x_0 \in \mathbb{R}^3, R, T > 0, m \in \mathbb{N}_0$ and a set of measure zero, S , which is contained in a compact subset of $B(x_0, R)$.

a. An *amenable blow-up rate of order m on $Q_T(S)$* is a function, $D(x, t) : \Omega_T(S) \rightarrow [0, \infty)$, which additionally satisfies,

- i. there exists $M_0 > 0$ so that $\sup_{0 < t \leq T} \|\log^{m+1} D(x, t)\|_{BMO} < M_0$,
- ii. there exist $M_1, M_2 > 0$ so that $1/M_1 \leq D(x, t)$ on $Q_T(S)$ and $D(x, t) \leq M_2$ on $B(x_0, R)^c \times (0, T]$.

b. A function $f(x, t) : Q_T(S) \rightarrow \mathbb{R}$ exhibits an *amenable blow-up rate of order 0 on $Q_T(S)$* if there exists an amenable blow-up rate of order 0, $D(x, t)$, on $Q_T(S)$, so that, for some $C_* > 1$,

$$D(x, t) \leq |f(x, t)| \leq C_* D(x, t) \quad \text{for all } (x, t) \in Q_T(S).$$

c. A function $f(x, t) : Q_T(S) \rightarrow \mathbb{R}$ exhibits an *amenable blow-up rate of order 1 on $Q_T(S)$* if there exists an amenable blow-up rate of order 1, $D(x, t)$, on $Q_T(S)$, so that, for some $C_* > 0, A \geq 1$, and a scalar function $\alpha : [0, T] \rightarrow [1, A]$,

$$D(x, t) \leq |f(x, t)| \leq C_* D(x, t)^{\alpha(t)} \quad \text{for all } (x, t) \in Q_T(S).$$

The condition (6.a.i) will prove crucial in establishing our estimates in Sect. 3. At face value, however, it is not clearly motivated. To address this we specify an expansive class of functions which are simultaneously amenable blow-up rates of order 0 and are reasonable blow-up scenarios given what is known about the structure of possible singularities in weak solutions of 3D NSE. In the following, $\mathbb{R}[x_1, x_2, x_3]$ denotes the ring of tri-variate polynomials with coefficients in \mathbb{R} .

Definition 7. Use the notations of Definition 6 and fix $d \in \mathbb{N}$. The function $D(x, t) : \Omega_T(S) \rightarrow [0, \infty)$ is an *algebraic blow-up rate of degree d on $Q_T(S)$* if there exist functions $\tau : (0, T] \rightarrow [0, \infty), \alpha : (0, T] \rightarrow (0, \infty)$, and $\rho(x, \cdot) : (0, T] \rightarrow \mathbb{R}[x_1, x_2, x_3]$ so that,

$$D(x, t) = \left(\frac{1}{|\rho(x, t)| + \tau(t)} \right)^{\alpha(t)} \quad \text{on } Q_T(S),$$

and we additionally have,

- i. for all $t \in (0, T], \rho(x, t)$ is a polynomial of degree less than or equal to d and the zeros of $\rho(x, T)$ are contained in S ,
- ii. there exists $A \geq 1$ so that α takes values in $[A^{-1}, A]$,
- iii. $\tau(t) > 0$ for $t \in (0, T)$ and vanishes as t approaches T ,
- iv. the condition (6.a.ii.) is satisfied.

Remark 8. We have presented these algebraic blow-up rates because they constitute a concrete class of amenable blow-up rates of order 0. To verify this, we check that condition (6.a.i.) is satisfied, which is clear if we first expand $\log D(x, t)$ as,

$$\log(D(x, t)) = -\alpha(t) \log(|\rho(x, t)| + \tau(t)),$$

and then observe that,

$$\max\{\log|\rho(x, t)|, \log\tau(t)\} \leq \log(|\rho(x, t)| + \tau(t)) \leq \log(2) + \max\{\log|\rho(x, t)|, \log\tau(t)\}.$$

Recalling the fact that BMO is a lattice, i.e. if $f, g \in BMO$, then,

$$\|\max\{f, g\}\|_{BMO} \leq 2(\|f\|_{BMO} + \|g\|_{BMO}),$$

we are able to conclude by applying the implication (1) in conjunction with Lemma 5.

Our main lemma contextualizes amenable blow-up rates to their application in Sect. 3. As presently defined, the blow-up rates require that $f(x, t)$ (as given in Definition 6) is bounded away from zero. To accommodate functions possibly *not* bounded away from zero we employ an auxiliary function,

$F = \sqrt{e_m + f^2}$, where e_m is determined by the order of the amenable blow-up rate in question. Based on the sign of f we define,

$$F_+ = \begin{cases} F, & \text{if } f \text{ is non-negative,} \\ 1, & \text{otherwise,} \end{cases} \quad F_- = \begin{cases} 1, & \text{if } f \text{ is non-negative,} \\ F, & \text{otherwise.} \end{cases} \tag{2}$$

The factorization $F^{\text{sign } f} = F_+(F_-)^{-1}$ will allow us to independently impose blow-up assumptions on a function's positive and negative components.

Lemma 9. *Fix $R_0 > 0$. Let $S' \subset \mathbb{R}^3$ be a set of Lebesgue measure zero for which $S' \cap B(0, R_0) \subset B(0, R'_0)$ where $0 < R'_0 < R_0$. Suppose the function $f : \Omega_T(S') \rightarrow \mathbb{R}$ satisfies $f \in L^\infty((0, T]; L^1(B(0, R_0)))$. Fix $\phi \in C_0^1(\mathbb{R}^3)$ so that ϕ is supported on $B(0, R)$ and let S be a set of measure zero satisfying $S' \cap B(0, R_0) \subset S \subset B(0, R_0)$.*

a. *If $f|_{Q_T(S)}$ exhibits an amenable blow-up of order 0 in $Q_T(S)$, then,*

$$\sup_{0 < t \leq T} \|\phi(x) \log |f(x, t)|\|_{BMO} < \infty.$$

b. *If $F_+|_{Q_T(S)}$ and $F_-|_{Q_T(S)}$ exhibit amenable blow-up rates of order 0 in $Q_T(S)$, then,*

$$\sup_{0 < t \leq T} \left\| \phi(x) \frac{f(x, t)}{F(x, t)} \log F(x, t) \right\|_{BMO} < \infty.$$

c. *If $F_+|_{Q_T(S)}$ and $F_-|_{Q_T(S)}$ exhibit amenable blow-up rates of order 1 in $Q_T(S)$, then,*

$$\sup_{0 < t \leq T} \left\| \phi(x) \frac{f(x, t)}{F(x, t)} \log \log F(x, t) \right\|_{BMO} < \infty.$$

Proof. (Part a.) Let $D(x, t)$ be the amenable blow-up rate exhibited by $f|_{Q_T(S)}$. We will freely reference the constants associated with D in the statement of Definition 6 and subsequently make the abbreviations $Q_T(S) = Q_T(0, R_0, S)$ and $B = B(0, R_0)$. Define an extension, \tilde{F} , of $f|_{Q_T(S)}$ to $\Omega_T(S)$ by $\tilde{F} = |f|$ on $Q_T(S)$ and $\tilde{F} = 1$ on $B^c \times (0, T]$. Set $\tilde{C}_* = \max\{C_*, M_2\}$ and $\tilde{D}(x, t) = \max\{D(x, t), M_1^{-1}\}$. These definitions ensure that, for all $(x, t) \in \Omega_T(S)$,

$$\frac{1}{\tilde{C}_*} \tilde{D}(x, t) \leq \tilde{F}(x, t) \leq \tilde{C}_* \tilde{D}(x, t),$$

and, taking logarithms,

$$\log(\tilde{D}(x, t)) - \log \tilde{C}_* \leq \log(\tilde{F}(x, t)) \leq \log(\tilde{D}(x, t)) + \log \tilde{C}_*.$$

Recalling the implication (1), we see that,

$$\|\log \tilde{F}(x, t)\|_{BMO} \leq C(M_0, \tilde{C}_*).$$

Because $|f| = \tilde{F}$ on the support of ϕ we have,

$$\begin{aligned} \|\phi \log |f|\|_{BMO} &= \|\phi \log \tilde{F}\|_{BMO} \\ &\leq C \|\nabla \phi\|_\infty \left(\|\log \tilde{F}\|_{BMO} + \left| \frac{1}{|B|} \int_B \log \tilde{F} \, dx \right| \right), \end{aligned}$$

where we have applied Lemma 4. The average appearing above is uniformly bounded in time; this follows from the facts that $M_1^{-1} \leq |f|$ and $f \in L^\infty((0, T]; L^1(B(0, R_0)))$.

(Part b.) We begin by remarking that, for all $y \in \mathbb{R}$,

$$\text{sign } y \log(\sqrt{1 + y^2}) - 1 \leq \frac{y}{\sqrt{1 + y^2}} \log(\sqrt{1 + y^2}) \leq \text{sign } y \log \sqrt{1 + y^2} + 1.$$

Taking $y = |f(x, t)|$, in light of implication (1) it is sufficient to work with the function $(\text{sign } f) \log F$, where F , and also F_+ and F_- , are defined as in the comments preceding the statement of the lemma. Those definitions were motivated by the fact that $F^{\text{sign } f} = F_+(F_-)^{-1}$ and, so,

$$\log(F^{\text{sign } f}) = \log(F_+) - \log(F_-).$$

Let G be the extension by 1 of $(fF^{-1} \log(F))|_{Q_T(S)}$ from $Q_T(S)$ to $\Omega_T(S)$ and adopt the notation from the proof of part a. of this lemma for definitions of functions analogous to those mentioned above where we designate by D_{\pm} the blow-up rates exhibited by F_{\pm} . Then, on $\Omega_T(S)$, we have,

$$\log(\tilde{F}_+) - \log(\tilde{F}_-) - 1 \leq G \leq \log(\tilde{F}_+) - \log(\tilde{F}_-) + 1,$$

and, therefore,

$$\|G\|_{BMO} \leq \|\log(\tilde{F}_+)\|_{BMO} + \|\log(\tilde{F}_-)\|_{BMO} + 2.$$

Appealing to part a. of the lemma, the dominating quantity is itself bounded uniformly in time. Extending this to the desired estimate proceeds directly,

$$\left\| \phi \frac{f}{F} \log(F) \right\|_{BMO} \leq C \|\nabla \phi\|_{\infty} \left(\|G\|_{BMO} + \left| \frac{1}{B} \int_B \log \tilde{G} \, dx \right| \right),$$

which, again, is finite by assumptions on f .

(Part c.) We begin similarly, noting,

$$(\text{sign } f) \log \log F - 1 \leq \frac{f}{F} \log \log F \leq (\text{sign } f) \log \log F + 1.$$

This leads us to consider,

$$\log \left((\log F)^{\text{sign } f} \right) = \log \left((\log F)_+ \right) + \log \left((\log F)_- \right),$$

where,

$$(\log F)_+ = \max \left\{ (\log F)^{\text{sign } f}, 1 \right\},$$

and,

$$(\log F)_- = \min \left\{ (\log F)^{\text{sign } f}, 1 \right\}.$$

It is easy to check that $(\log F)_{\pm} = \log(F_{\pm})$, and, therefore,

$$\frac{f}{F} \log \log F = \log \log(F_+) - \log \log(F_-).$$

A further observation is that, taking $D_{\pm}(x, t)$ to be the amenable blow-up rates exhibited by F_{\pm} , and setting $k_{\pm}^* = \log(A_{\pm}M_{1,\pm} + \log C_{*,\pm})$ (these are constants associated with D_{\pm} as in Definition 6.a.), we have for $(x, t) \in Q_T$,

$$\log \log(C_* D_{\pm}(x, t)^{\alpha_{\pm}(t)}) \leq \log \log(D_{\pm}(x, t)) + k_{\pm}^*,$$

and, recalling the comparability condition from Definition 6.c., this implies that,

$$\log \log(D_{\pm}(x, t)) \leq F_{\pm} \leq \log \log(D_{\pm}(x, t)) + k_{\pm}^*.$$

The conclusion now follows in the same fashion laid out in part b.; we omit the details. □

3. Main Results

Weak solutions to the 3D incompressible Navier–Stokes equations are functions which satisfy (distributionally) the following system of PDEs,

$$\begin{cases} \partial_t u + (u \cdot \nabla)u = -\nabla p + \nu \Delta u, \\ \nabla \cdot u = 0; \quad u(x, 0) = u_0 \in H, \end{cases} \tag{3}$$

where H is the L^2 closure of the divergence free test functions and the initial datum is understood in the sense of weak continuity (cf. [7] for details). The evolution of the vorticity, $\omega = \nabla \times u$, is of special interest to us and satisfies,

$$\begin{cases} \partial_t \omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u + \nu \Delta \omega, \\ \nabla \cdot \omega = 0. \end{cases} \tag{4}$$

For simplicity, we consider a weak solution on $\mathbb{R}^3 \times (0, \infty)$, evolving from the initial data u_0 . We also require that the initial vorticity, $\omega_0 = \nabla \times u_0$, is in $L^1 \cap L^2$.

The standard regularity results for weak solutions (cf. [7]) consist of the *a priori* bounds,

$$\sup_{0 < t < T'} \|u\|_{L^2(\mathbb{R}^3)}^2 < \infty \quad \text{and} \quad \int_0^{T'} \|\nabla u\|_{L^2(\mathbb{R}^3)}^2 dt < \infty,$$

for any $T' > 0$. In addition, since $\omega_0 \in L^1$, a result from [6] ensures that,

$$\sup_{0 \leq t \leq T'} \|\omega\|_{L^1(\mathbb{R}^3)} < \infty.$$

Note that since $\omega_0 \in L^2$, our weak solution locally-in-time coincides with the smooth solution; let T be the first (possible) singular time.

Fix a ‘macro-scale,’ $R_0 > 0$, with the property that the intersection of $B(0, R_0) \times \{T\}$ with the singular set at time T is nonempty. Fix $0 < \epsilon < R_0$. Our estimates are intended for integrals over the spatial set $B(0, R_0 - \epsilon)$ and localization is achieved via multiplying equation (4) by a smooth cut-off function ψ taking values in $[0, 1]$ and satisfying,

$$\text{supp } \psi \subset B(0, R_0), \quad \psi = 1 \text{ on } B(0, R_0 - \epsilon), \quad \text{and} \quad \frac{|\nabla \psi|}{\psi^\rho} \leq \frac{c}{\epsilon},$$

where $\rho \in (0, 1)$ is fixed. Instead of studying the evolution of $|\omega|$ and $|\omega_k|$ directly we introduce an auxiliary function. This approach is an adaptation of that taken by Constantin in [6]. Define $q(y) = \sqrt{1 + y^2} : \mathbb{R} \rightarrow \mathbb{R}$ and let $w_k = q(\omega_k) : \mathbb{R}^3 \rightarrow \mathbb{R}$. From these definitions it is immediate that,

$$|\omega_k| \leq w_k, \quad -1 \leq q'(\omega_k) = \frac{\omega_k}{w_k} \leq 1, \quad \text{and} \quad 0 < q''(\omega_k) = \frac{1}{w_k^3} \leq 1. \tag{5}$$

For convenience we recall the following elementary facts (for $y \geq 1$),

$$0 \leq -\log''(y) \leq \log'(y) \leq 1, \quad y \log' y = 1, \quad \text{and} \quad 0 = \log'(y) + y \log''(y). \tag{6}$$

To allow ω_k to have varying sign we define the functions $w_{k,+}$ and $w_{k,-}$ in the same manner as (2). Recall that $\Lambda_t(y) = \{x : |\omega(x, t)| \geq y\}$ and note that throughout the remainder of this paper c_1 denotes a fixed constant taken to be greater than one. The role of c_1 lies in specifying the threshold for the super-level sets $\Lambda_t(y)$ as fractions of the supremum norm of the modulus of the vorticity at the time t , i.e. $y = c_1^{-1} \|\omega(\cdot, t)\|_\infty$.

We include two theorems, one each for the case of solutions possessing vorticity components which exhibit amenable blow-up rates of order 0 and of order 1. The proof of the order 0 case extends easily to the case of blow-ups of order 1 and the proof of the second result is accordingly terse.

Theorem 10. *Let u be a Leray solution to (3) on $\mathbb{R}^3 \times (0, \infty)$, and suppose additionally that $\omega_0 = \nabla \times u_0 \in L^1 \cap L^2$. Denote by T the first singular time and by S the singular set of ω at time T . Fix positive values R_0 and ϵ so that $\emptyset \neq S \cap B(0, R_0) \subset B(0, R_0 - \epsilon)$ —i.e. there are singular points in $B(0, R_0)$ but all such points are in $B(0, R_0 - \epsilon)$ —and let $Q_T(S) = Q_T(0, R_0, S)$.*

- (i) *If $w_{k,+}|_{Q_T(S)}$ and $w_{k,-}|_{Q_T(S)}$ each exhibit amenable blow-up rates of order 0 in $Q_T(S)$, then there exists a positive value M_k for which,*

$$\sup_{t \in [0, T]} \int_{B(0, R_0 - \epsilon)} |\omega_k(x, t)| \log(\sqrt{1 + |\omega_k(x, t)|^2}) \, dx < M_k.$$

- (ii) *Let c_1 be a fixed constant which is greater than 1. If, for all k , the premises of part (i) are satisfied, then there exists a positive constant M_0 so that, for $t \in [0, T]$,*

$$\text{Vol}\left(\Lambda_t\left(\frac{1}{c_1} \|\omega(t)\|_{L^\infty(B(0, R_0 - \epsilon))}\right)\right) \leq \frac{M_0}{\|\omega(t)\|_{L^\infty(B(0, R_0 - \epsilon))} \log(1 + \|\omega(t)\|_{L^\infty(B(0, R_0 - \epsilon))})}.$$

Remark 11. As will be seen, our energy inequality-type method depends heavily on classical techniques which leads us to consider local-in-time smooth solutions. It is possible to extend the estimates to *some* weak solutions by considering a sequence of smooth approximations, e.g. the retarded mollifiers from [1], but, as our estimates hinge on assumptions beyond the initial data, u_0 , we would have to ensure these are met by a convergent sub-sequence of approximate solutions.

Proof. In virtue of our smoothness assumption, the evolution of $\psi w_k \log(w_k)$ can be established from the evolution of ω_k by first writing,

$$\begin{aligned} \partial_t(\psi w_k \log w_k) &= \psi q'(\omega_k)(\log w_k + w_k \log' w_k)(\partial_t \omega_k) \\ &= \psi q'(\omega_k)(\log w_k + w_k \log' w_k)(\nu \Delta \omega_k - (u \cdot \nabla) \omega_k + (\omega \cdot \nabla) u_k), \end{aligned}$$

which leads to (noting tacit summation over terms involving indices other than k),

$$\begin{aligned} \partial_t(\psi w_k \log w_k) - \nu \psi \Delta w_k (\log w_k + w_k \log' w_k) + \nu \psi q''(\omega_k) (\partial_i \omega_k)^2 (\log w_k + w_k \log' w_k) \\ = \psi (\omega \cdot \nabla) u_k q'(\omega_k) (\log w_k + w_k \log' w_k) - \psi (u \cdot \nabla) w_k (\log w_k + w_k \log' w_k). \end{aligned}$$

By integrating in space and time and dropping the positive quantity involving q'' from the left hand side, we obtain the following energy inequality-type estimate,

$$\begin{aligned} \int \psi(t) w_k(t) \log w_k(t) \, dx - \nu \int_0^t \int \psi \Delta w_k (\log w_k + w_k \log' w_k) \, dx \, ds \\ \leq \int_0^t \int \psi (\omega \cdot \nabla) u_k q'(\omega_k) (\log w_k + w_k \log' w_k) \, dx \, ds \\ - \int_0^t \int \psi (u \cdot \nabla) w_k (\log w_k + w_k \log' w_k) \, dx \, ds \\ + \int \psi w_{k,0} \log w_{k,0} \, dx. \end{aligned} \tag{7}$$

The properties in (6) enable several key cancellations. For the dissipative terms, integration by parts reveals that,

$$\begin{aligned} \int_0^t \int (\partial_j^2 w_k) \psi \log w_k \, dx \, ds &= \int_0^t \int \left(w_k (\partial_j^2 \psi) \log w_k + w_k (\partial_j \psi) (\partial_j w_k) \log' w_k \right) \, dx \, ds \\ &\quad - \int_0^t \int (\partial_j w_k)^2 \psi \log' w_k \, dx \, ds, \end{aligned}$$

and,

$$\begin{aligned} \int_0^t \int (\partial_j^2 w_k) \psi w_k \log' w_k \, dx \, ds &= - \int_0^t \int w_k (\partial_j \psi) (\partial_j w_k) \log' w_k \, dx \, ds \\ &\quad - \int_0^t \int (\partial_j w_k)^2 \psi \left(\log' w_k + w_k \log'' w_k \right) \, dx \, ds. \end{aligned}$$

A cancellation occurs upon adding the above equations leaving us with,

$$\begin{aligned} -\nu \int_0^t \int (\partial_j^2 w_k) \psi \left(\log w_k + w_k \log' w_k \right) \, dx \, ds &= \nu \int_0^t \int (\partial_j w_k)^2 \psi \left(2 \log' w_k + w_k \log'' w_k \right) \, dx \, ds \\ &\quad - \nu \int_0^t \int w_k (\partial_j^2 \psi) \log w_k \, dx \, ds. \end{aligned}$$

Again noting (6), the integrand of the first term is positive and, therefore, can be dropped from the left hand side of estimate (7). The second term can be dominated by an *a priori* finite quantity arising from the standard energy inequality for weak solutions in conjunction with the fact that our spatial integral is over a set of finite measure. More precisely,

$$\begin{aligned} \int_0^t \int w_k (\partial_j^2 \psi) \log w_k \, dx \, ds &\leq C \int_0^T \int_{B(0, R_0)} w_k^2 \, dx \, ds \\ &= C \int_0^T \int_{B(0, R_0)} |\omega_k|^2 \, dx \, ds + T |B(0, R_0)|. \end{aligned} \tag{8}$$

The integrals arising from the transport term in (4) also enjoy substantial cancellations,

$$\begin{aligned} \int_0^t \int \psi \, u \cdot \nabla w_k \left(\log w_k + w_k \log' w_k \right) \, dx \, ds &= \int_0^t \int \psi \, u_j (\partial_j w_k) w_k \log' w_k \, dx \, ds \\ &\quad - \int_0^t \int \psi \, u_j w_k (\partial_j w_k) \log' w_k \, dx \, ds \end{aligned}$$

$$\begin{aligned}
 & - \int_0^t \int u_j w_k \log w_k \partial_j \psi \, dx \, ds \\
 & = - \int_0^t \int u_j w_k \log w_k \partial_j \psi \, dx \, ds.
 \end{aligned}$$

Noting that $\log(y) \leq 4y^{1/4}$,

$$\begin{aligned}
 \left| \int_0^t \int u_j w_k \log w_k \partial_j \psi \, dx \, ds \right| & \leq C \int_0^T \int (|\partial_j \psi|^{1/4} |u_j|) (|\partial_j \psi|^{1/2} |w_k|) |w_k \partial_j \psi|^{1/4} \, dx \, ds \\
 & \leq C \int_0^T \|\psi^{\rho/4} u\|_4 \|w_k\|_{L^1(B(0,R_0))}^{1/4} \|w_k\|_{L^2(B(0,R_0))} \\
 & \leq C \sup_{0 < t \leq T} \|w_k\|_{L^1(B(0,R_0))}^{1/4} \int_0^T \|\nabla(\psi^{\rho/4} u)\|_2 \|w_k\|_{L^2(B(0,R_0))},
 \end{aligned}$$

where we have used Hölder's inequality and the Sobolev inequality. From here and in light of (8), a commutator estimate on the gradient of the localized velocity allows the extension of the above estimate to one in terms of *a priori* finite quantities.

The integral on the last line of the right hand side of (7) is finite by our assumptions on the initial data.

At this point, based on (7), we have established,

$$\int (\psi w_k \log w_k)(t) \, dx \leq \left| \int_0^t \int \psi \, \omega \cdot \nabla u_k \, q'(\omega_k) \left(\log w_k + w_k \log' w_k \right) \, dx \, ds \right| + R,$$

where R is comprised of those *a priori* bounded quantities accumulated in the preceding estimates. Noting that,

$$\psi q'(\omega_k) w_k \log' w_k \leq 1,$$

an *a priori* bound follows for that part of the as of yet unbounded quantity leaving us with,

$$\int (\psi w_k \log w_k)(t) \, dx \leq \left| \int_0^t \int \psi \, \omega \cdot \nabla u_k \, \frac{\omega_k}{w_k} \log w_k \, dx \, ds \right| + R.$$

The \mathcal{H}^1 -BMO duality and the Div-Curl lemma justify the following chain of inequalities,

$$\begin{aligned}
 & \left| \int_0^t \int \left(\omega \cdot \nabla u_k \right) \left(\psi \frac{\omega_k}{w_k} \log w_k \right) \, dx \, ds \right| \\
 & \leq \int_0^T \|\omega \cdot \nabla u_k\|_{\mathcal{H}^1} \left\| \psi \frac{\omega_k}{w_k} \log w_k \right\|_{BMO} \, ds
 \end{aligned}$$

$$\begin{aligned} &\leq \left(\sup_{0 < t \leq T} \left\| \psi \frac{\omega_k}{w_k} \log w_k \right\|_{BMO} \right) \int_0^T \|\omega \cdot \nabla u_k\|_{\mathcal{H}^1} ds \\ &\leq \left(\sup_{0 < t \leq T} \left\| \psi \frac{\omega_k}{w_k} \log w_k \right\|_{BMO} \right) \left(\int_0^T \|\omega\|_2^2 dt \right)^{1/2} \left(\int_0^T \|\nabla u\|_2^2 dt \right)^{1/2}. \end{aligned}$$

By Lemma 9 and the standard regularity of Leray weak solutions all of the above are finite and we have thus established that, for all $0 < t \leq T$, $\|\psi w_k \log w_k(t)\|_{L^1(\mathbb{R}^3)}$ is majorized by time-independent *a priori* bounded quantities, the sum of which we label M_k . This completes our proof of part (i) of the theorem.

Part (ii) of the theorem is proven in two steps. For the first, let,

$$\lambda(t) = \frac{1}{c_1} \|\omega(t)\|_{L^\infty(B(0, R_0 - \epsilon))},$$

and observe that, for any $x \in B(0, R_0 - \epsilon)$ where $|\omega(x, t)| \geq \lambda(t)$, direct computation affirms that,

$$1 \leq \frac{c_1 |\omega(x, t)| [\log(c_1) + \log(1 + |\omega(x, t)|)]}{\|\omega(t)\|_{L^\infty(B(0, R_0 - \epsilon))} \log(1 + \|\omega(t)\|_{L^\infty(B(0, R_0 - \epsilon))}}.$$

This allows us to estimate the volume of the relevant super-level set of $|\omega|$ at time t ,

$$\begin{aligned} \text{Vol} \left(\Lambda_t \left(\frac{1}{c_1} \|\omega(t)\|_{L^\infty(B(0, R_0 - \epsilon))} \right) \right) &\leq \frac{c_1 \log(c_1) \|\omega(t)\|_{L^1(\{|\omega(x, t)| \geq \lambda(t)\} \cap B(0, R_0 - \epsilon))}}{\|\omega(t)\|_{L^\infty(B(0, R_0 - \epsilon))} \log(1 + \|\omega(t)\|_{L^\infty(B(0, R_0 - \epsilon))})} \\ &\quad + \frac{c_1 \|\omega(t) \log(1 + |\omega(t)|)\|_{L^1(\{|\omega(x, t)| \geq \lambda(t)\} \cap B(0, R_0 - \epsilon))}}{\|\omega(t)\|_{L^\infty(B(0, R_0 - \epsilon))} \log(1 + \|\omega(t)\|_{L^\infty(B(0, R_0 - \epsilon))})} \\ &\leq \frac{KI_0(t)}{\|\omega(t)\|_{L^\infty(B(0, R_0 - \epsilon))} \log(1 + \|\omega(t)\|_{L^\infty(B(0, R_0 - \epsilon))})}, \end{aligned}$$

where we have set,

$$I_0(t) = \int_{\{|\omega(x, t)| \geq \lambda(t)\} \cap B(0, R_0 - \epsilon)} |\omega(x, t)| \log(1 + |\omega(x, t)|) dx,$$

and have introduced a time-independent constant K which depends on the fixed values c_1 and R_0 as well as the *a priori* finite quantity,

$$\sup_{0 \leq t \leq T} \|\omega(\cdot, t)\|_{L^1(\{|\omega(x, t)| \geq \lambda(t)\} \cap B(0, R_0 - \epsilon))}.$$

The second step ensures we can control $I_0(t)$ in terms of the finite bounds appearing in part (i) of the theorem. Tacitly summing over j , we have,

$$I_0(t) \leq C \int_{B(0, R_0 - \epsilon)} |\omega_j| \log(1 + \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}) dx.$$

An explicit reduction illustrates the argument (for simplicity we take $j = 1$ and integrals over the indicated sets intersected with $B(0, R_0 - \epsilon)$),

$$\begin{aligned} \int |\omega_1| \log(1 + \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}) dx &\leq \int_{\omega_1^2 \leq \omega_2^2 + \omega_3^2} \sqrt{\omega_2^2 + \omega_3^2} \log(1 + \sqrt{2(\omega_2^2 + \omega_3^2)}) dx \\ &\quad + \int_{\omega_1^2 > \omega_2^2 + \omega_3^2} |\omega_1| \log(1 + \sqrt{2}|\omega_1|) dx. \end{aligned}$$

Applying the same reasoning to the first integral above and then repeating for all values of j eventually yields,

$$I_0(t) \leq C \int w_i \, dx + C \int w_i \log w_i \, dx \leq CM_i.$$

□

The energy inequality-type construction used to prove the previous theorem also works if we substitute $\log^m \omega_k$ (as defined in Sect. 2) in place of $\log w_k$. This allows the application of amenable blow-up rates of order 1 in conjunction with Lemma 9.c. To ensure things are meaningful, we modify our definition of q so that $q(y) = \sqrt{e + y^2}$ (so, now, $w_k = \sqrt{e + \omega_k^2}$) and refer to (2) to define $w_{k,+}$ and $w_{k,-}$.

Theorem 12. *Let u be a Leray solution to (3) on $\mathbb{R}^3 \times (0, \infty)$, and suppose additionally that $\omega_0 = \nabla \times u_0 \in L^1 \cap L^2$. Denote by T the first singular time and by S the singular set of ω at time T . Fix positive values R_0 and ϵ so that $\emptyset \neq S \cap B(0, R_0) \subset B(0, R_0 - \epsilon)$ —i.e. there are singular points in $B(0, R_0)$ but all such points are in $B(0, R_0 - \epsilon)$ —and let $Q_T(S) = Q_T(0, R_0, S)$.*

(i) *If $w_{k,+}|_{Q_T(S)}$ and $w_{k,-}|_{Q_T(S)}$ each exhibit amenable blow-up rates of order 1 in $Q_T(S)$, then there exists a positive value M_k so that,*

$$\sup_{t \in [0, T]} \int_{B(0, R_0 - \epsilon)} |\omega_k(x, t)| \log \log (\sqrt{e + |\omega_k(x, t)|^2}) \, dx < M_k.$$

(ii) *Let c_1 be a fixed constant which is greater than 1. If, for all k , the premises of part (i) are satisfied, then there exists a positive value M_0 so that, for $t \in [0, T]$,*

$$\text{Vol} \left(\Lambda_t \left(\frac{1}{c_1} \|\omega(t)\|_{L^\infty(B(0, R_0 - \epsilon))} \right) \right) \leq \frac{M_0}{\|\omega(t)\|_{L^\infty(B(0, R_0 - \epsilon))} \log \log (e + \|\omega(t)\|_{L^\infty(B(0, R_0 - \epsilon))})}.$$

Proof. Multiplying equation (4) by $\psi q'(w_k)(\log \log w_k + w_k(\log \log)'w_k)$ we obtain the evolution of $\psi w_k \log \log w_k$. The point-wise estimates in (6) adapt directly to the function $\log \log(y)$ (indeed, they adapt to any number of such self-compositions of the logarithm) and, after integrating in space and time, all of the estimates and cancellations from the previous proof—except those involving the vortex stretching term—can be duplicated directly. We thus obtain, denoting by R some *a priori* finite quantity,

$$\int (\psi w_k \log \log w_k)(t) \, dx \leq \left| \int_0^t \int \psi \omega \cdot \nabla u_k \frac{\omega_k}{w_k} \log \log w_k \, dx \, ds \right| + R, \tag{9}$$

and, using Lemma 9.c., we are able to pull out of the integral the uniformly-in-time bounded quantity,

$$\sup_t \left\| \psi \frac{\omega_k}{w_k} \log \log w_k \right\|_{BMO},$$

and conclude exactly as in the previous proof. □

Remark 13. Theorems 10 and 12 apply to any local, spatially-algebraic blow-up scenario in which the degrees of the polynomials stay *uniformly bounded* near the (possible) singular time. Geometrically, this corresponds to the number of vortex filaments being uniformly bounded as the flow approaches the singular time. This is somewhat unsatisfactory since, *a priori*, one can not rule out a scenario in which the number of coherent structures runs off to infinity. The technical reason behind this restriction is that the bound on $\|\log |P|\|_{BMO}$ blows up as the degree of P, d , goes to infinity. Fortunately, the bound on $\|\log |P|\|_{BMO}$ is *linear in d* (cf. [9]). Consequently, although the bound on the distribution function, i.e. on the total volume of the super-level sets, will blow up with d , at least in the case of comparable volumes, the bound on the volume of a single vortex filament will still be sub-critical (which suffices due to the local nature of the argument).

Acknowledgments. Z.B. acknowledges the support of the *Virginia Space Grant Consortium* via the Graduate Research Fellowship; Z.G. acknowledges the support of the *Research Council of Norway* via the Grant 213474/F20, and the *National Science Foundation* via the Grant DMS 1212023.

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(accepted: October 1, 2013; published online: December 15, 2013)