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Local analyticity radii of solutions to the 3D Navier–Stokes equations with locally analytic forcing

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Abstract

We introduce a new method for establishing local analyticity and estimating the local analyticity radius of a solutions to the 3D Navier–Stokes equations at interior points. The approach is based on rephrasing the problem in terms of second order parabolic systems which are then estimated using the mild solution approach. The estimates agree with the global analyticity radius from [16] up to a logarithm.

MSC: 35Q30; 76D05

Keywords: Navier–Stokes equations; Strong solutions; Analyticity

1. Introduction

In this paper we consider the analyticity radius of solutions to the 3D Navier–Stokes system

$$\begin{aligned} \partial_t u - \Delta u &= -u \cdot \nabla u - \nabla p + f \quad &&\text{in } \mathbb{R}^3 \times (0, T) \\ \nabla \cdot u &= 0 &&\text{in } \mathbb{R}^3 \times (0, T), \end{aligned} \tag{3D NSE}$$

where the force f is locally analytic. In the first result we provide a lower bound for the analyticity radius of strong solutions to 3D NSE evolving from initial data in L^q where $q > 3$. In the second result we estimate the analyticity radius of locally smooth solutions from below in terms

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of strictly local quantities including the local Reynolds number. This is formulated in purely local terms and applies to the boundary value Navier–Stokes problem yielding estimates for the local analyticity radius of solutions at interior points.

The main motivation for studying the analyticity radii of solutions to viscous fluid models is their connection to the dissipative length scales from turbulence theories [9,11–13,19,20,32]. At and below the dissipative scale, inertial range cascade dynamics break down and frictional effects become the dominant influence on energy transport dynamics. In analytic solutions this shift is visible as the exponential fall off of the Fourier spectrum at frequencies beyond the inverse of the analyticity radius. Another motivation for studying the analyticity radius reflects its applicability to geometric measure-type regularity criteria (see [6,15]). In particular, the radius of spatial analyticity has been identified as the scale of local, anisotropic diffusion in the vorticity formulation of the 3D NSE.

Classical analyticity results for solutions to 3D NSE can be found in [2,14,23,31]. A pioneering work in the area of estimating analyticity radii was carried out by Foias and Temam in [10] using Fourier techniques and Gevrey spaces in an L^2 setting (see also [8,33]). Related results in L^p spaces were obtained in [27,28]. This approach has subsequently been revisited using more modern techniques in a variety of function spaces (see, e.g., [1,3–5,21,30,33,34]). An alternative approach to the problem in L^p spaces where $p \in (3, \infty]$ was developed in [16,18,24] and is carried out entirely in physical space. This strategy is more tailored to accommodate local settings than the Fourier space techniques, a fact shown in [17] where it is applied to solutions of a non-linear heat equation at interior points of bounded domains. Gevrey space techniques have also been applied to study the decay of analyticity radii associated with solutions to the 3D Euler Equations evolving from analytic initial data on the whole space (see [26]) and domains with boundaries (see [25]).

Most available analyticity results for solutions to 3D NSE are global and require the forcing term f to be analytic with a uniform analyticity radius. In this paper we provide a new method which allows us to estimate the analyticity radius in local contexts using the mild solution approach as in [7,22]. In contrast to the global argument of [16], our strategy involves solving second order parabolic systems. The resulting argument is simple and provides a new approach at interior points of bounded domains which avoids the need for complicated recursive estimates like those found in [23].

The paper is organized as follows. In Section 2 we formulate our local assumptions on the forcing and state the main results, Theorems 2.1 and 2.3. The approximation schemes for our inductive arguments are presented in Section 3 where we also prove Theorem 2.1. Section 4 is dedicated to proving Theorem 2.3.

2. Statement of main results

Our first result provides a local lower bound on the analyticity radius of a flow subjected to real-analytic forcing f possessing a possibly non-uniform analyticity radius. With $\lambda_f(x,t)$ denoting the radius of spatial analyticity of f at (x,t), let

$$\lambda_{f,T}(x) = \inf_{t\in(0,T]} \lambda_f(x,t).$$

For x_* in $\mathbb{R}^3$ fixed, denote by B_* the ball of radius $r_* = \lambda_{f,T}(x_*)/2$ centered at x_*. Then, if $x \in 2B_*$ it follows that $\lambda_{f,T}(x) \geq 2r_* - |x - x_*|$, and, consequently, $f(x,t)$ is the restriction to $\mathbb{R}^3$ of a function $F(x, y, t) + iG(x, y, t)$ which is defined and complex analytic on the domain

$$\Omega_{f,T}(x_*) = \{x + iy \in \mathbb{C}^3 : x \in 2B_*,\ |y| < 2r_* - |x - x_*|\},$$

provided $t \in (0, T]$.

Let ψ be a non-negative test function, which is supported on $2B_*$, evaluates to 1 on B_*, is radially non-increasing in $|x - x_*|$, and additionally satisfies the estimates

$$|\psi(x)| \le \frac{2r_* - |x - x_*|}{r_*},$$

and

$$||\nabla\psi||_\infty \le \frac{C}{r^*}.$$

Let $\alpha \in \mathbb{R}^3$ be such that $(x, \alpha\psi(x)t, t)$ belongs to the domain of analyticity of $F + iG$ provided $t \le r_*^2$; denote the set of all permissible vectors α by S_f. Let $F_\alpha(x, t) = F(x, \alpha\psi(x)t, t)$ and $G_\alpha(x, t) = G(x, \alpha\psi(x)t, t)$.

Theorem 2.1. *Assume that u_0 and f are divergence free vector fields, that $u_0,\ f \in L^q(\mathbb{R}^3)$ for some $q > 3$, and that f agrees with the restriction of an analytic function $F + iG$ to $2B_*$. Let u denote Kato's strong solution on $(0, T_0)$ where $T_0 > 0$ (cf. [22, Theorem 1]). Fix $r > 2q/(q - 3)$ and assume that for some positive time $T < T_0$ we have*

$$M_f = \sup_{\alpha\in S_f} \Big(\sup_{0<\tau<T} \|F_\alpha(\tau)\|_{L^q(\mathbb{R}^3)} + \sup_{0<\tau<T} \|G_\alpha(\tau)\|_{L^q(\mathbb{R}^3)}\Big) < \infty, \tag{2.1}$$

and

$$M_0 = ||u_0||_{L^q} + C\,T^{(r-2)/2r} \left(\int_0^T ||\nabla u(\tau)||_{L^q}^r \, d\tau \right)^{\frac{1}{r}} < \infty. \tag{2.2}$$

Let

$$T_1 \le \frac{1}{C} \min\left\{T,\ r_*^2,\ \left(\frac{1}{M_0}\right)^{2q/(q-3)},\ \frac{||u_0||_{L^q}}{M_f}\right\}.$$

Then, for any $t \in (0, T_1)$, the solution $u(t)$ agrees with the restriction to B_* of a function $u(x, y, t) + iv(x, y, t)$ which is defined and analytic in the region

$$\Omega_*(t) = \left\{x + iy \in \mathbb{C}^3 :\ x \in B_*,\ |y| < \frac{\sqrt{t}}{4\,C_0}\right\},$$

for a universal constant C_0.

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Remark 2.2.

(i) In order to motivate the inclusion of the quantity M_0 note that, up to an ϵ, it is controlled by $||u_0||_q$. To see this let $\epsilon > 0$ be given and note that, since u agrees with Kato's strong solution from [22], we have

$$\left(\int_\epsilon^T ||\nabla u(\tau)||_{L^q}^r \, d\tau\right)^{1/r} \le \left(\int_\epsilon^T \left(||u_0||_q \, t^{-1/2}\right)^r d\tau\right)^{1/r}$$
$$\le C\,(r-2)^{-1/r}\,\epsilon^{(2-r)/(2r)}\,||u_0||_q.$$

As $q \to \infty$ we observe that $r \to 2$ indicating that for large q the $L^r(0,T;L^q(\mathbb{R}^3))$ norm of ∇u, when finite, is close to its $L^2(0,T;L^q(\mathbb{R}^3))$ norm.

(ii) The theorem leads to a lower bound for the local analyticity radius of $u(x_*, T_1)$, namely

$$\lambda_u(x_*, T_1) \gtrsim \min\left\{r_*, \left(\frac{1}{M_0}\right)^{q/(q-3)}, \left(\frac{||u_0||_{L^q}}{M_f}\right)^{1/2}\right\}.$$

This is consistent with results obtained in [16] for globally analytic forcing. In fact, it is possible to show that the local estimate for the radius is a logarithmic correction of the global estimate in [16].

The second result considers a system evolving from localized initial data $\mu_0 = \phi\, u_0$ where ϕ is a non-negative test function compactly supported on the ball $4B_*$. In this context lower bounds on the analyticity radius are established in terms of purely local quantities. Note that this result is valid if, in the definition of 3D NSE, we replace $\mathbb{R}^3$ with any domain properly containing $4B_*$.

Theorem 2.3. *Assume that u_0 is a divergence free vector field with support in $4B_*$, that f is divergence free and contained in $L^q(4B_*)$ for some $q > 3$, and that f agrees with the restriction of an analytic function $F + iG$ to $2B_*$. Suppose that u solves 3D NSE with the data u_0 and f and is smooth on $(0, T_0) \times 4B_*$ for some $T_0 > 0$. Fix $r > 2q/(q-3)$ and assume that for a time $T \in (0, T_0)$ we have*

$$M_{f,\mathrm{loc}} = C \sup_{\alpha \in S_f} \Big(\sup_{0<\tau<T} \|F_\alpha(\tau)\|_{L^q(4B_*)} + \sup_{0<\tau<T} \|G_\alpha(\tau)\|_{L^q(4B_*)}\Big) < \infty,$$

and

$$M_{\mathrm{loc}} = C \sup_{0<\tau<T} ||u(\tau)||_{L^q(4B_*)} + C \sup_{0<\tau<T} ||p(\tau)||_{L^{q/2}(4B_*)}^{1/2}$$
$$+ C\, T^{(r-2)/(2r)} \left(\int_0^T ||\nabla u||_{L^q(4B_*)}^r\right)^{\frac{1}{r}} < \infty.$$

Let

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$$T_2 \le \frac{1}{C}\min\left\{T,\ r_*^2,\ \left(\frac{1}{M_{\text{loc}}}\right)^{2q/(q-3)},\ \frac{M_{\text{loc}}}{M_{f,\text{loc}}}\right\}.$$

Then, at any time $t \in (0, T_2)$, the solution u agrees with the restriction to B_* of a function $u(x, y, t) + iv(x, y, t)$ which is analytic in the region,

$$\Omega_*(t) = \{x + iy \in \mathbb{C}^3 :\ x \in B_*,\ |y| < \frac{\sqrt{t}}{4C_0}\},$$

for a universal constant C_0.

3. Locally analytic forcing and global dependence on initial data

This section contains the proof of Theorem 2.1. In order to study the analyticity radii of u at points in B_* we locally extend elements of a classical approximation scheme for 3D NSE into $\mathbb{C}^3$. Let

$$u^{(0)} = p^{(0)} = 0,$$

and, for $n \ge 1$, successively determine $u^{(n)}$ by solving the systems

$$\begin{aligned}
&\partial_t u^{(n)} - \Delta u^{(n)} = -u^{(n-1)} \cdot \nabla u^{(n-1)} - \nabla p^{(n-1)} + f && \text{in } \mathbb{R}^3 \times (0, T)\\
&\nabla \cdot u^{(n)} = 0 && \text{in } \mathbb{R}^3 \times (0, T)\\
&\Delta p^{(n-1)} = -\partial_i \partial_j \big(u_i^{(n-1)} u_j^{(n-1)}\big) && \text{in } \mathbb{R}^3 \times (0, T)\\
&u^{(n)}(\cdot, 0) = u_0(\cdot) && \text{in } \mathbb{R}^3.
\end{aligned}$$

It is well known that $u^{(n)}$ converge to a solution of 3D NSE in $C([0, T); L^q(\mathbb{R}^3))$ for sufficiently small values of T (see [22]). Let $\alpha \in S_f$, i.e. $(x, \alpha\psi(x)t, t)$ is in the domain of analyticity of $F + iG$. The real analyticity of f and the smoothing properties of the heat and Poisson equations imply that $u^{(n)}$ and $p^{(n)}$ are restrictions to $\mathbb{R}^3$ of functions $U^{(n)} + iV^{(n)}$ and $P^{(n)} + i\Pi^{(n)}$ which are analytic on the same complex strip as $-u^{(n-1)} \cdot \nabla u^{(n-1)} - \nabla p^{(n-1)} + f$. Since the initial iteration was obtained using $u^{(0)} = 0$, we see for any $t \in (0, T]$ that, $U^{(1)}(x + iy, t) + iV^{(1)}(x + iy, t)$ and $P^{(1)}(x + iy, t) + i\Pi^{(1)}(x + iy, t)$ are analytic on $\Omega_{f,T}(x_*)$ and, reasoning inductively, this observation extends to all approximate solutions. Moreover, at points in $\Omega_{f,T}(x_*) \times (0, T)$, we have

$$\begin{aligned}
&\partial_t U^{(n)} - \Delta U^{(n)} = -U^{(n-1)} \cdot \nabla U^{(n-1)} + V^{(n-1)} \cdot \nabla V^{(n-1)} - \nabla P^{(n-1)} + F\\
&\partial_t V^{(n)} - \Delta V^{(n)} = -U^{(n-1)} \cdot \nabla V^{(n-1)} - V^{(n-1)} \cdot \nabla U^{(n-1)} - \nabla \Pi^{(n-1)} + G\\
&\Delta P^{(n-1)} = -\partial_i \partial_j \big(U_i^{(n-1)} U_j^{(n-1)} - V_i^{(n-1)} V_j^{(n-1)}\big)\\
&\Delta \Pi^{(n-1)} = -2\partial_i \partial_j \big(U_i^{(n-1)} V_j^{(n-1)}\big)\\
&\nabla \cdot U^{(n)} = \nabla \cdot V^{(n)} = 0.
\end{aligned}$$

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Note that $(x, \alpha t\psi(x), t)$ is in the domain of analyticity of $U^{(n)} + iV^{(n)}$ and $P^{(n)} + i\Pi^{(n)}$. Let $U_\alpha^{(n)}(x,t) = U^{(n)}(x, \alpha t\psi(x), t)$ with analogous definitions for $V_\alpha^{(n)}$, $P_\alpha^{(n)}$, and $\Pi_\alpha^{(n)}$ and for the forcing terms.

Next, we derive evolution equations for $U_\alpha^{(n)}$ and $V_\alpha^{(n)}$ and equations for $P_\alpha^{(n)}$ and $\Pi_\alpha^{(n)}$. Since the Cauchy–Riemann system

$$\begin{aligned} \partial_i U^{(n)} &= \partial_i^* V^{(n)}, \qquad \partial_i^* U^{(n)} = -\partial_i V^{(n)}, \\ \partial_i P^{(n)} &= \partial_i^* \Pi^{(n)}, \qquad \partial_i^* P^{(n)} = -\partial_i \Pi^{(n)} \end{aligned}$$

is satisfied on $\Omega_{f,T}(x_*)$ (here ∂_i^* denotes the partial derivative in the i-th complex variable), the time derivatives of $U_\alpha^{(n)}$ and $V_\alpha^{(n)}$ satisfy

$$\begin{aligned} \partial_t U_\alpha^{(n)}(x,t) &= (\partial_t U^{(n)})(x, \alpha t\psi(x), t) - \alpha_l \psi(x)(\partial_l V^{(n)})(x, \alpha t\psi(x), t) \\ \partial_t V_\alpha^{(n)}(x,t) &= (\partial_t V^{(n)})(x, \alpha t\psi(x), t) + \alpha_l \psi(x)(\partial_l U^{(n)})(x, \alpha t\psi(x), t), \end{aligned}$$

while for the spatial derivatives we have

$$\begin{aligned} \partial_j U_\alpha^{(n)}(x,t) &= \partial_j U^{(n)}(x, \alpha t\psi(x), t) - \alpha_l t \partial_j \psi(x)(\partial_l V^{(n)})(x, \alpha t\psi(x), t) \\ \partial_j V_\alpha^{(n)}(x,t) &= \partial_j V^{(n)}(x, \alpha t\psi(x), t) + \alpha_l t \partial_j \psi(x)(\partial_l U^{(n)})(x, \alpha t\psi(x), t). \end{aligned}$$

If

$$|\alpha| t \le \frac{r_*}{C_0}, \tag{3.1}$$

where C_0 is a sufficiently large positive constant, we may algebraically solve the above system in order to obtain

$$\begin{aligned} (\partial_j U^{(n)})(x, \alpha t\psi(x), t) &= b_{jk}^{11} \partial_k U_\alpha^{(n)} + b_{jk}^{12} \partial_k V_\alpha^{(n)} \\ (\partial_j V^{(n)})(x, \alpha t\psi(x), t) &= b_{jk}^{21} \partial_k U_\alpha^{(n)} + b_{jk}^{22} \partial_k V_\alpha^{(n)}, \end{aligned}$$

where the coefficient functions b_{jk}^{lm}, are smooth, non-negative, and bounded, with gradients supported in $2B_*$. By the condition (3.1), all the first order derivatives of b_{ij}^{mn} are uniformly bounded. Extending this reasoning to second derivatives yields

$$\begin{aligned} (\partial_i \partial_j U^{(n)})(x, \alpha t\psi(x), t) &= b_{il}^{11} \partial_l \Big(b_{jm}^{11} \partial_m U_\alpha^{(n)}(x,t) + b_{jm}^{12} \partial_m V_\alpha^{(n)}(x,t) \Big) \\ &\quad + b_{il}^{12} \partial_l \Big(b_{jm}^{21} \partial_m U_\alpha^{(n)}(x,t) + b_{jm}^{22} \partial_m V_\alpha^{(n)}(x,t) \Big) \\ (\partial_i \partial_j V^{(n)})(x, \alpha t\psi(x), t) &= b_{il}^{21} \partial_l \Big(b_{jm}^{11} \partial_m U_\alpha^{(n)}(x,t) + b_{jm}^{12} \partial_m V_\alpha^{(n)}(x,t) \Big) \\ &\quad + b_{il}^{22} \partial_l \Big(b_{jm}^{21} \partial_m U_\alpha^{(n)}(x,t) + b_{jm}^{22} \partial_m V_\alpha^{(n)}(x,t) \Big). \end{aligned}$$

Again using the Cauchy–Riemann system as well as repeated applications of the chain rule, we obtain other useful formulas

$$\begin{aligned}(\Delta U^{(n)})(x, \alpha t\psi(x), t) &= \Delta U^{(n)}_\alpha(x,t) + \partial_l(a^{11}_{lk}\partial_k U^{(n)}_\alpha(x,t)) + \partial_l(a^{12}_{lk}\partial_k V^{(n)}_\alpha(x,t)) \\ &\quad + \widetilde{c}^{11}_l \partial_l U^{(n)}_\alpha(x,t) + \widetilde{c}^{12}_l \partial_l V^{(n)}_\alpha(x,t), \\ (\Delta V^{(n)})(x, \alpha t\psi(x), t) &= \Delta V^{(n)}_\alpha(x,t) + \partial_l(a^{21}_{lk}\partial_k U^{(n)}_\alpha(x,t)) + \partial_l(a^{22}_{lk}\partial_k V^{(n)}_\alpha(x,t)) \\ &\quad + \widetilde{c}^{21}_l \partial_l U^{(n)}_\alpha(x,t) + \widetilde{c}^{22}_l \partial_l V^{(n)}_\alpha(x,t),\end{aligned}$$

where the coefficient functions are supported on $2B_*$ with

$$||a^{lm}_{jk}||_\infty \lesssim |\alpha|\, t\, |\nabla\psi|, \tag{3.2}$$

and

$$||\widetilde{c}^{lm}_j||_\infty,\ ||\nabla a^{lm}_{jk}||_\infty \lesssim \frac{|\alpha|\, t\, |\nabla\psi|}{r_*}. \tag{3.3}$$

The smallness condition (3.1) ensures that

$$||a^{lm}_{jk}||_\infty \le \frac{C}{C_0},$$

and

$$||\widetilde{c}^{lm}_j||_\infty \le \frac{C}{r_* C_0}.$$

The representations involving spatial derivatives hold for any conjugate pair of analytic functions on $\Omega_{f,T}(x_*)$; in particular, they hold for $P^{(n)}$, $P^{(n)}_\alpha$, $\Pi^{(n)}$, and $\Pi^{(n)}_\alpha$. They also lead to evolution equations for $U^{(n)}_\alpha$ and $V^{(n)}_\alpha$, namely,

$$\begin{aligned}&\partial_t U^{(n)}_\alpha - \Delta U^{(n)}_\alpha \\ &\quad = -U^{(n-1)}_{\alpha,l}\big(b^{11}_{lk}\partial_k U^{(n-1)}_\alpha + b^{12}_{lk}\partial_k V^{(n-1)}_\alpha\big) + V^{(n-1)}_{\alpha,l}\big(b^{21}_{lk}\partial_k U^{(n-1)}_\alpha + b^{22}_{lk}\partial_k V^{(n-1)}_\alpha\big) \\ &\qquad - b^{11}_{lk}\partial_k P^{(n-1)}_\alpha - b^{12}_{lk}\partial_k \Pi^{(n-1)}_\alpha + \alpha_l c^{11}_{lm}\partial_m U^{(n)}_\alpha + \alpha_l c^{12}_{lm}\partial_m V^{(n)}_\alpha \\ &\qquad + \partial_l(a^{11}_{lk}\partial_k U^{(n)}_\alpha) + \partial_l(a^{12}_{lk}\partial_k V^{(n)}_\alpha) + \widetilde{c}^{11}_l\partial_l U^{(n)}_\alpha + \widetilde{c}^{12}_l\partial_l V^{(n)}_\alpha + F_\alpha,\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}&\partial_t V^{(n)}_\alpha - \Delta V^{(n)}_\alpha \\ &\quad = -V^{(n-1)}_{\alpha,l}\big(b^{11}_{lk}\partial_k U^{(n-1)}_\alpha + b^{12}_{lk}\partial_k V^{(n-1)}_\alpha\big) - U^{(n-1)}_{\alpha,l}\big(b^{21}_{lk}\partial_k U^{(n-1)}_\alpha + b^{22}_{lk}\partial_k V^{(n-1)}_\alpha\big) \\ &\qquad - b^{21}_{lk}\partial_k P^{(n-1)}_\alpha - b^{22}_{lk}\partial_k \Pi^{(n-1)}_\alpha + \alpha_l c^{21}_{lm}\partial_m U^{(n)}_\alpha + \alpha_l c^{22}_{lm}\partial_m V^{(n)}_\alpha \\ &\qquad + \partial_l(a^{21}_{lk}\partial_k U^{(n)}_\alpha) + \partial_l(a^{22}_{lk}\partial_k V^{(n)}_\alpha) + \widetilde{c}^{21}_l\partial_l U^{(n)}_\alpha + \widetilde{c}^{22}_l\partial_l V^{(n)}_\alpha + G_\alpha,\end{aligned} \tag{3.5}$$

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where c^{ij}_{lm} are smooth, bounded, and supported in $2B_*$. We similarly see that $P^{(n)}_\alpha$ and $\Pi^{(n)}_\alpha$ satisfy

$$\begin{aligned}-\Delta P^{(n)}_\alpha &= \partial_l(a^{11}_{lk}\partial_k P^{(n)}_\alpha) + \partial_l(a^{12}_{lk}\partial_k \Pi^{(n)}_\alpha) + \tilde{c}^{11}_l \partial_l P^{(n)}_\alpha + \tilde{c}^{12}_l \partial_l \Pi^{(n)}_\alpha \\ &\quad + b^{11}_{ik}\partial_k\Big(b^{11}_{jl}\partial_l\big(U^{(n)}_{\alpha,i}U^{(n)}_{\alpha,j} - V^{(n)}_{\alpha,i}V^{(n)}_{\alpha,j}\big) + b^{12}_{jl}\partial_l\big(2U^{(n)}_{\alpha,i}V^{(n)}_{\alpha,j}\big)\Big) \\ &\quad + b^{12}_{ik}\partial_k\Big(b^{21}_{jl}\partial_l\big(U^{(n)}_{\alpha,i}U^{(n)}_{\alpha,j} - V^{(n)}_{\alpha,i}V^{(n)}_{\alpha,j}\big) + b^{22}_{jl}\partial_l\big(2U^{(n)}_{\alpha,i}V^{(n)}_{\alpha,j}\big)\Big), \end{aligned} \tag{3.6}$$

and

$$\begin{aligned}-\Delta \Pi^{(n)}_\alpha &= \partial_l(a^{21}_{lk}\partial_k P^{(n)}_\alpha) + \partial_l(a^{22}_{lk}\partial_k \Pi^{(n)}_\alpha) + \tilde{c}^{21}_l \partial_l P^{(n)}_\alpha) + \tilde{c}^{22}_l \partial_l \Pi^{(n)}_\alpha \\ &\quad + b^{21}_{ik}\partial_k\Big(b^{11}_{jl}\partial_l\big(U^{(n)}_{\alpha,i}U^{(n)}_{\alpha,j} - V^{(n)}_{\alpha,i}V^{(n)}_{\alpha,j}\big) + b^{12}_{jl}\partial_l\big(2U^{(n)}_{\alpha,i}V^{(n)}_{\alpha,j}\big)\Big) \\ &\quad + b^{12}_{ik}\partial_k\Big(b^{21}_{jl}\partial_l U^{(n)}_{\alpha,i}U^{(n)}_{\alpha,j} - V^{(n)}_{\alpha,i}V^{(n)}_{\alpha,j}\big) + b^{12}_{jl}\partial_l\big(2U^{(n)}_{\alpha,i}V^{(n)}_{\alpha,j}\big)\Big). \end{aligned} \tag{3.7}$$

Estimates for the solutions of these systems are contained in the following two lemmas.

Lemma 3.1. *Let u, p, and T_1 be as in the statement of Theorem 2.1. If $|\alpha| T_1^{1/2} \le 1/(2C_0)$ for a universal constant C_0 then*

$$||P^{(n)}_\alpha(t)||_{L^{q/2}},\ ||\Pi^{(n)}_\alpha(t)||_{L^{q/2}} \le C\big(||U^{(n)}_\alpha(t)||^2_{L^q} + ||V^{(n)}_\alpha(t)||^2_{L^q}\big), \tag{3.8}$$

and

$$\begin{aligned}&||\nabla P^{(n)}_\alpha(t)||_{L^{q/2}},\ ||\nabla \Pi^{(n)}_\alpha(t)||_{L^{q/2}} \\ &\quad \le C\big(||U^{(n)}_\alpha(t)||_{L^q} + ||V^{(n)}_\alpha(t)||_{L^q}\big)\big(||\nabla U^{(n)}_\alpha(t)||_{L^q} + ||\nabla V^{(n)}_\alpha(t)||_{L^q}\big),\end{aligned} \tag{3.9}$$

for all $n \in \mathbb{N}$ and $t \in (0, T_1)$.

Proof. Using the elliptic systems (3.6) and (3.7) we expand $P^{(n)}_\alpha(x,t)$ in terms of singular integral operators as

$$\begin{aligned}P^{(n)}_\alpha(x,t) &= \int_{\mathbb{R}^3} \frac{1}{|x-y|}\Delta P^{(n)}_\alpha(y,t)\,dy \\ &= \int_{\mathbb{R}^3} \frac{1}{|x-y|}\Big(\partial_l(a^{11}_{lk}\partial_k P^{(n)}_\alpha) + \partial_l(a^{12}_{lk}\partial_k \Pi^{(n)}_\alpha) + \tilde{c}^{11}_l \partial_l P^{(n)}_\alpha + \tilde{c}^{12}_l \partial_l \Pi^{(n)}_\alpha\Big)(y,t)\,dy\end{aligned}$$

$$+\int_{\mathbb{R}^3} \frac{1}{|x-y|}\left(b_{ik}^{11}\partial_k\left(b_{jl}^{11}\partial_l\left(U_{\alpha,i}^{(n)}U_{\alpha,j}^{(n)}-V_{\alpha,i}^{(n)}V_{\alpha,j}^{(n)}\right)+b_{jl}^{12}\partial_l\left(2U_{\alpha,i}^{(n)}V_{\alpha,j}^{(n)}\right)\right)\right)(y,t)\,dy$$

$$+\int_{\mathbb{R}^3} \frac{1}{|x-y|}\left(b_{ik}^{12}\partial_k\left(b_{jl}^{21}\partial_l\left(U_{\alpha,i}^{(n)}U_{\alpha,j}^{(n)}-V_{\alpha,i}^{(n)}V_{\alpha,j}^{(n)}\right)+b_{jl}^{22}\partial_l\left(2U_{\alpha,i}^{(n)}V_{\alpha,j}^{(n)}\right)\right)\right)(y,t)\,dy.$$

As t is fixed, it is suppressed throughout the remainder of this section. We label the first term on the far right side by I and the sum of the remaining two terms by J. Integrating by parts, I can be written so that no derivatives fall on $P_\alpha^{(n)}$ and $\Pi_\alpha^{(n)}$, i.e.

$$\begin{aligned} I &= P.V.\int_{\mathbb{R}^3} \partial_k\partial_l\left(\frac{1}{|x-y|}\right)\left(a_{lk}^{11}P_\alpha^{(n)}+a_{lk}^{12}\Pi_\alpha^{(n)}\right)(y)\,dy \\ &\quad+\int_{\mathbb{R}^3} \partial_l\left(\frac{1}{|x-y|}\right)\left((\partial_k a_{lk}^{11})P_\alpha^{(n)}+(\partial_k a_{lk}^{12})\Pi_\alpha^{(n)}\right)(y)\,dy \\ &= I_1+I_2. \end{aligned}$$

By the Calderon–Zygmund theorem (see [35]) we have

$$\begin{aligned} ||I_1||_{L^{q/2}} &\le C\left(||a_{lk}^{11}P_\alpha^{(n)}||_{L^{q/2}}+||a_{lk}^{12}\Pi_\alpha^{(n)}||_{L^{q/2}}\right) \\ &\le C||a_{lk}^{ij}||_{L^\infty}\left(||P_\alpha^{(n)}||_{L^{q/2}}+||\Pi_\alpha^{(n)}||_{L^{q/2}}\right) \\ &\le \frac{C\,|\alpha|\,t}{r_*}\left(||P_\alpha^{(n)}||_{L^{q/2}}+||\Pi_\alpha^{(n)}||_{L^{q/2}}\right). \end{aligned}$$

Provided $|\alpha|t$ is sufficiently small we infer

$$||I_1||_{L^{q/2}} \le \frac{1}{4}\left(||P_\alpha^{(n)}||_{L^{q/2}}+||\Pi_\alpha^{(n)}||_{L^{q/2}}\right).$$

For I_2, since the coefficient functions a_{lj}^{ij} are compactly supported and their gradients are controlled by a multiple of r_*^{-1}, by making C_0 large and $|\alpha|t$ correspondingly small we obtain

$$||I_2||_{L^{q/2}} \le \frac{1}{4}\left(||P_\alpha^{(n)}||_{L^{q/2}}+||\Pi_\alpha^{(n)}||_{L^{q/2}}\right).$$

To estimate J we integrate by parts so that no gradients fall on the components of $U_\alpha^{(n)}$ and $V_\alpha^{(n)}$. Then, applying the Calderon–Zygmund and Hardy–Littlewood–Sobolev inequalities yields

$$||J||_{L^{q/2}} \le C\left(||U_\alpha^{(n)}||_{L^q}^2+||V_\alpha^{(n)}||_{L^q}^2\right).$$

Taken together and noting an analogous argument holds for $\Pi_\alpha^{(n)}$, these estimates justify (3.8). For the gradients of $P_\alpha^{(n)}$ and $\Pi_\alpha^{(n)}$, a similar argument and the fact that the integral formulas are of convolution type lead to the bound (3.9). □

Lemma 3.2. *If $u,\ p$, and T_1 are as in the statement of Theorem 2.1 and $|\alpha| T_1^{1/2} \le 1/(2C_0)$ for a universal constant C_0, then*

$$\sup_{0<t<T} \|U_\alpha^{(n)}(t)\|_{L^q} + \sup_{0<t<T} \|V_\alpha^{(n)}(t)\|_{L^q} \le 5\, M_0 \tag{3.10}$$

for all $T \in (0, T_1)$ and $n \in \mathbb{N}$.

Proof. We inductively estimate the L^q norms of $U_\alpha^{(n+1)}$ and $V_\alpha^{(n+1)}$. Let $\widetilde{U}_\alpha^{(k)} = U_\alpha^{(k)} - u$. We show that if, for all $T \in (0, T_1)$ and some $n \in \mathbb{N}$, the inequalities

$$\sup_{0<t<T} \|\widetilde{U}_\alpha^{(n)}(t)\|_{L^q} + \sup_{0<t<T} \|V_\alpha^{(n)}(t)\|_{L^q} \le 4\, M_0, \tag{3.11}$$

and

$$T^{(r-2)/(2r)} \left(\left(\int_0^T \|\nabla \widetilde{U}_\alpha^{(n)}(t)\|_{L^q}^r \, dt \right)^{\frac{1}{r}} + \left(\int_0^T \|\nabla V_\alpha^{(n)}(t)\|_{L^q}^r \, dt \right)^{\frac{1}{r}} \right) \le 4\, M_0, \tag{3.12}$$

are satisfied, then also

$$\sup_{0<t<T} \|\widetilde{U}_\alpha^{(n+1)}(t)\|_{L^q} + \sup_{0<t<T} \|V_\alpha^{(n+1)}(t)\|_{L^q} \le 4\, M_0, \tag{3.13}$$

and

$$T^{(r-2)/(2r)} \left(\left(\int_0^T \|\nabla \widetilde{U}_\alpha^{(n+1)}(t)\|_{L^q}^r \, dt \right)^{\frac{1}{r}} + \left(\int_0^T \|\nabla V_\alpha^{(n+1)}(t)\|_{L^q}^r \, dt \right)^{\frac{1}{r}} \right) \le 4\, M_0, \tag{3.14}$$

provided $|\alpha|$ is controlled in terms of T_1 and r_*. Since u represents Kato's strong L^q solution we have $||u(t)||_{L^q} \le c_0\, M_0$ for $0 < t < T_0$ where T_0 is appropriately small. Upon establishing (3.13), the conclusion (3.10) clearly follows.

To avoid repetition we focus on $\widetilde{U}_\alpha^{(n+1)}$ noting that the argument for $V_\alpha^{(n+1)}$ is similar. Using Duhamel's principle we obtain

$$\begin{aligned}
\widetilde{U}_\alpha^{(n+1)}(t) = & \int_0^t e^{(t-\tau)\Delta} \big(F_\alpha - f\big)(\tau)\, d\tau \\
& + \int_0^t e^{(t-\tau)\Delta} \big(\alpha_l c_{lm}^{11} \partial_m \widetilde{U}_\alpha^{(n+1)} + \alpha_l c_{lm}^{11} \partial_m u + \alpha_l c_{lm}^{12} \partial_m V_\alpha^{(n+1)}\big)(\tau)\, d\tau \\
& + \int_0^t e^{(t-\tau)\Delta} \big(\partial_l (a_{lk}^{11} \partial_k \widetilde{U}_\alpha^{(n+1)}) + \partial_l (a_{lk}^{11} \partial_k u) + \partial_l (a_{lk}^{12} \partial_k V_\alpha^{(n+1)})\big)(\tau)\, d\tau
\end{aligned}$$

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$$+\int_0^t e^{(t-\tau)\Delta}\big(\widetilde{c}_l^{11}\partial_l\widetilde{U}_\alpha^{(n+1)}+\widetilde{c}_l^{11}\partial_l u+\widetilde{c}_l^{12}\partial_l V_\alpha^{(n+1)}\big)(\tau)\,d\tau$$

$$-\int_0^t e^{(t-\tau)\Delta}\Big(U_{\alpha,l}^{(n)}\big(b_{lk}^{11}\partial_k U_\alpha^{(n)}+b_{lk}^{12}\partial_k V_\alpha^{(n)}\big)-u_l\partial_l u$$

$$-V_{\alpha,l}^{(n)}\big(b_{lk}^{21}\partial_k U_\alpha^{(n)}+b_{lk}^{22}\partial_k V_\alpha^{(n)}\big)\Big)(\tau)\,d\tau$$

$$-\int_0^t e^{(t-\tau)\Delta}\big(b_{lk}^{11}\partial_k P_\alpha^{(n)}-\partial_l p+b_{lk}^{12}\partial_k\Pi_\alpha^{(n)}\big)(\tau)\,d\tau, \qquad (3.15)$$

which is valid for $t \in (0, T_0)$.

Our first task is to obtain bounds for $\|\widetilde{U}_\alpha^{(n+1)}(t)\|_{L^q}$ and $\|V_\alpha^{(n+1)}(t)\|_{L^q}$ where $t \in (0, T)$ with $T < T_0$. The first term from (3.15) is bounded as

$$\Big\|\int_0^t e^{(t-\tau)\Delta}(F_\alpha(\tau)-f)\,d\tau\Big\|_{L^q} \le C\,T\Big(\sup_{0<\tau<T}\|F_\alpha(\tau)\|_{L^q}+\sup_{0<\tau<T}\|f(\tau)\|_{L^q}\Big).$$

Estimates for the remaining terms involve an exponent

$$r \in \big(2q/(q-3), \infty\big).$$

The L^q estimates for the second and third terms of (3.15) follow from standard techniques (cf. [22]). The terms with one gradient are dealt with using Young's convolution inequality and Hölder's inequality and assuming $T^{1/2} \le r_*$. This gives

$$\Big\|\int_0^t e^{(t-\tau)\Delta}\alpha_l c_{lm}^{11}\partial_m\widetilde{U}_\alpha^{(n+1)}(\tau)\,d\tau\Big\|_{L^q} \le C|\alpha|T^{(r-1)/r}\Big(\int_0^T\|\nabla\widetilde{U}_\alpha^{(n+1)}(t)\|_{L^q}^r\,dt\Big)^{\frac{1}{r}},$$

while similar estimates hold for the terms involving $V_\alpha^{(n)}$ and u. For the terms with two derivatives we move a derivative to the Gaussian kernel (this also introduces lower order terms but these can be shown to satisfy similar bounds) and obtain

$$\Big\|\int_0^t \partial_l e^{(t-\tau)\Delta}a_{lk}^{11}\partial_k\widetilde{U}_\alpha^{(n+1)}(\tau)\,d\tau\Big\|_{L^q} \le C|\alpha|T^{(r-1)/r}\Big(\int_0^T\|\nabla\widetilde{U}_\alpha^{(n+1)}(t)\|_{L^q}^r\,dt\Big)^{\frac{1}{r}}.$$

Again we note that analogous estimates hold for terms involving $V_\alpha^{(n)}$ and u.

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The first bilinear term in (3.15) is estimated as

$$\Big\| \int_0^t e^{(t-\tau)\Delta} U^{(n)}_{\alpha,l} b^{11}_{lk} \partial_k U^{(n)}_\alpha(\tau)\, d\tau \Big\|_{L^q}$$

$$\le C\, T^{(r-1)/r-3/(2q)} \sup_{0<\tau<T} \|U^{(n)}_\alpha(\tau)\|_{L^q} \Big(\int_0^t \|\nabla U^{(n)}_\alpha(\tau)\|^r_{L^q}\, d\tau \Big)^{\frac{1}{r}}, \tag{3.16}$$

with similar estimates holding for the remaining bilinear terms. To deal with the pressure terms from (3.15) we use (3.9) to obtain

$$\Big\| \int_0^t e^{(t-\tau)\Delta} b^{11}_{lk} \partial_k P^{(n)}_\alpha(\tau)\, d\tau \Big\|_{L^q}$$

$$\le C\, T^{(r-1)/r-3/(2q)} \big(\sup_{0<\tau<T} ||U^{(n)}_\alpha(\tau)||_{L^q} + \sup_{0<\tau<T} ||V^{(n)}_\alpha(\tau)||_{L^q} \big)$$

$$\times \Big(\Big(\int_0^T \|\nabla U^{(n)}_\alpha(\tau)\|^r_{L^q}\, d\tau \Big)^{\frac{1}{r}} + \Big(\int_0^T \|\nabla V^{(n)}_\alpha(\tau)\|^r_{L^q}\, d\tau \Big)^{\frac{1}{r}} \Big).$$

We thus obtain the L^q estimate

$$\begin{aligned}
&\|\widetilde{U}^{(n+1)}_\alpha(t)\|_{L^q} + \|V^{(n+1)}_\alpha(t)\|_{L^q} \\
&\le C\, T \big(\sup_{0<\tau<T} \|F_\alpha(\tau)\|_{L^q} + \sup_{0<\tau<T} \|f(\tau)\|_{L^q} + \sup_{0<\tau<T} \|G_\alpha(\tau)\|_{L^q} \big) \\
&\quad + C\,|\alpha|\, T^{(r-1)/r} \Big(\Big(\int_0^T \|\nabla \widetilde{U}^{(n+1)}_\alpha(t)\|^r_{L^q}\, dt \Big)^{\frac{1}{r}} + \Big(\int_0^T \|\nabla V^{(n+1)}_\alpha(t)\|^r_{L^q}\, dt \Big)^{\frac{1}{r}} \Big) \\
&\quad + C\, T^{(r-1)/r-3/(2q)} \big(\sup_{0<\tau<T} ||U^{(n)}_\alpha(\tau)||_{L^q} + \sup_{0<\tau<T} ||V^{(n)}_\alpha(\tau)||_{L^q} \big) \\
&\quad \times \Big(\Big(\int_0^T \|\nabla U^{(n)}_\alpha(\tau)\|^r_{L^q}\, d\tau \Big)^{\frac{1}{r}} + \Big(\int_0^T \|\nabla V^{(n)}_\alpha(\tau)\|^r_{L^q}\, d\tau \Big)^{\frac{1}{r}} \Big) \\
&\quad + C|\alpha| T^{1/2} M_0 + C\, T^{(q-3)/(2q)}\, M_0^2.
\end{aligned} \tag{3.17}$$

To complete the inductive argument we need L^r–L^q estimates for $\nabla \widetilde{U}^{(n+1)}_\alpha$ and $\nabla V^{(n+1)}_\alpha$. These are obtained by applying spatial gradients to the integral formulas for $\widetilde{U}^{(n+1)}_\alpha$ and $\nabla V^{(n+1)}_\alpha$ and then applying the $L^r(0,T;L^q(\mathbb{R}^3))$ norm to each term. For the terms involving the forcing we obtain bounds such as

$$\left(\int_0^T \left\| \int_0^t e^{(t-\tau)\Delta} \nabla F_\alpha(\tau)\, d\tau \right\|_{L^q}^r dt\right)^{\frac{1}{r}} \le C\, T^{\frac{2+r}{2r}} \big(\sup_{0<\tau<T} \|F_\alpha(\tau)\|_{L^q} + \sup_{0<\tau<T} \|f(\tau)\|_{L^q}\big). \tag{3.18}$$

The linear terms from (3.15) involving two gradients all satisfy estimates analogous to

$$\left(\int_0^T \left\| \int_0^t \nabla e^{(t-\tau)\Delta} \alpha_l c_{lm}^{11} \partial_m \widetilde{U}_\alpha^{(n+1)}(\tau)\, d\tau \right\|_{L^q}^r dt\right)^{\frac{1}{r}} \le C\, |\alpha|\, T^{1/r} \sup_{0<t<T} \|\widetilde{U}_\alpha^{(n+1)}(t)\|_{L^q}. \tag{3.19}$$

Now we turn to the critical linear terms which involve three gradients. For these we use the maximal L^r–L^q-regularity of the heat kernel (cf. [29]) to obtain

$$\left(\int_0^T \left\| \int_0^t \nabla \partial_l e^{(t-\tau)\Delta} a_{lk}^{11} \partial_k \widetilde{U}_\alpha^{(n+1)}(\tau)\, d\tau \right\|_{L^q}^r dt\right)^{\frac{1}{r}} \le C\, |\alpha|\, T^{1/2} \left(\int_0^T \|\partial_k \widetilde{U}_\alpha^{(n+1)}(t)\|_{L^q}^r\, dt\right)^{\frac{1}{r}}, \tag{3.20}$$

where we have used the assumptions (3.2) and $T \le r_*^2$. Note that similar estimates hold for the other critical terms.

For the bilinear terms, we use the requirement $r > 2q/(q-3)$ and obtain

$$\begin{aligned} &\left(\int_0^T \left\| \int_0^t \nabla e^{(t-\tau)\Delta} U_{\alpha,l}^{(n)} b_{lk}^{11} \partial_k U_\alpha^{(n)}(\tau)\, d\tau \right\|_{L^q}^r dt\right)^{\frac{1}{r}} \\ &\quad \le C\, T^{(q-3)/(2q)} \sup_{0<\tau<T} \|U_\alpha^{(n)}(\tau)\|_{L^q} \left(\int_0^T \|\nabla U_\alpha\|_{L^q}^r\, dt\right)^{\frac{1}{r}}, \end{aligned} \tag{3.21}$$

with analogous estimates holding for other bilinear terms. Finally, the pressure terms all satisfy estimates identical to

$$\begin{aligned} &\left(\int_0^T \left\| \int_0^t \nabla e^{(t-\tau)\Delta} b_{lk}^{11} \partial_k P_\alpha^{(n)}(\tau)\, d\tau \right\|_{L^q}^r dt\right)^{1/r} \\ &\le C\, T^{(q-3)/(2q)} \big(\sup_{0<\tau<T} ||U_\alpha^{(n)}(\tau)||_{L^q} + \sup_{0<\tau<T} ||V_\alpha^{(n)}(\tau)||_{L^q}\big) \\ &\quad \times \left(\left(\int_0^T \|\nabla U_\alpha^{(n)}(\tau)\|_{L^q}^r\, d\tau\right)^{\frac{1}{r}} + \left(\int_0^T \|\nabla V_\alpha^{(n)}(\tau)\|_{L^q}^r\, d\tau\right)^{\frac{1}{r}} \right). \end{aligned}$$

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Combining these bounds we obtain the L^r–L^q estimate

$$\begin{aligned}
&\Big(\int_0^T \|\nabla \widetilde{U}_\alpha^{(n+1)}(t)\|_{L^q}^r\,dt\Big)^{\frac{1}{r}} + \Big(\int_0^T \|\nabla V_\alpha^{(n+1)}(t)\|_{L^q}^r\,dt\Big)^{\frac{1}{r}} \\
&\quad\le C\,T^{(2+r)/(2r)}\big(\sup_{0<\tau<T}\|F_\alpha(\tau)\|_{L^q} + \sup_{0<\tau<T}\|f(\tau)\|_{L^q} + \sup_{0<\tau<T}\|G_\alpha(\tau)\|_{L^q}\big) \\
&\qquad + C\,|\alpha|\,T^{1/r}\Big(\sup_{0<\tau<T}\|\widetilde{U}_\alpha^{(n+1)}(\tau)\|_{L^q} + \sup_{0<\tau<T}\|u(\tau)\|_{L^q} + \sup_{0<\tau<T}\|V_\alpha^{(n+1)}(\tau)\|_{L^q}\Big) \\
&\qquad + C\,|\alpha|\,T^{1/2}\Big(\Big(\int_0^T \|\nabla \widetilde{U}_\alpha^{(n+1)}(t)\|_{L^q}^r\,dt\Big)^{\frac{1}{r}} + \Big(\int_0^T \|\nabla u(t)\|_{L^q}^r\,dt\Big)^{\frac{1}{r}} \\
&\qquad + \Big(\int_0^T \|\nabla V_\alpha^{(n+1)}(t)\|_{L^q}^r\,dt\Big)^{\frac{1}{r}}\Big) \\
&\qquad + C\,T^{(q-3)/(2q)}\big(\sup_{0<\tau<T}||U_\alpha^{(n)}(\tau)||_{L^q} + \sup_{0<\tau<T}||u(\tau)||_{L^q} + \sup_{0<\tau<T}||V_\alpha^{(n)}(\tau)||_{L^q}\big) \\
&\qquad \times\Big(\Big(\int_0^T \|\nabla U_\alpha^{(n)}(\tau)\|_{L^q}^r\,d\tau\Big)^{\frac{1}{r}} + \Big(\int_0^T \|\nabla u(t)\|_{L^q}^r\,dt\Big)^{\frac{1}{r}} + \Big(\int_0^T \|\nabla V_\alpha^{(n)}(\tau)\|_{L^q}^r\,d\tau\Big)^{\frac{1}{r}}\Big).
\end{aligned} \tag{3.22}$$

This inequality is improved using two assumptions, namely that $|\alpha|$ satisfies

$$|\alpha|T^{1/2} \le \frac{1}{2\,C_0}, \tag{3.23}$$

for a large enough constant C_0 and that the inductive hypothesis, (3.12), holds. We thus have

$$\begin{aligned}
&\Big(\int_0^T \|\nabla \widetilde{U}_\alpha^{(n+1)}(t)\|_{L^q}^r\,dt\Big)^{\frac{1}{r}} + \Big(\int_0^T \|\nabla V_\alpha^{(n+1)}(t)\|_{L^q}^r\,dt\Big)^{\frac{1}{r}} \\
&\quad\le C\,|\alpha|\,T^{1/r}\,M_0 + T^{(2+r)/(2r)}\,M_f + T^{(q-3)/(2q)-(r-2)/(2r)}\,M_0^2 + T^{(2-r)/2r}\,M_0 \\
&\qquad + C\,|\alpha|\,T^{1/r}\Big(\sup_{0<\tau<T}\|\widetilde{U}_\alpha^{(n+1)}\|_{L^q} + \sup_{0<\tau<T}\|V_\alpha^{(n+1)}\|_{L^q}\Big).
\end{aligned} \tag{3.24}$$

To conclude our inductive argument we return to (3.17) which we modify using (3.11), (3.23), and (3.24) to obtain

$$\sup_{0<t<T}\|\widetilde{U}_\alpha^{(n+1)}(t)\|_{L^q} + \sup_{0<t<T}\|V_\alpha^{(n+1)}(t)\|_{L^q} \le 2M_0 + T\,M_f + T^{(q-3)/(2q)}\,M_0^2.$$

Inequality (3.13) follows directly from the assumption on T_1, i.e. that

$$T_1 \leq \min\left\{r_*^2, \left(\frac{1}{C_0\, M_0}\right)^{2q/(q-3)}, \frac{M_0}{M_f}\right\}. \tag{3.25}$$

Using (3.13) and (3.24) we conclude that (3.14) holds whenever $T \leq T_1$. □

For convenience we recall a lemma which is stated and proven in [16].

Lemma 3.3. *(See [16, Lemma 2.4].) Let $\mathcal{F}$ be the set of all functions f which are analytic in an open set $\Omega \subseteq \mathbb{C}^d$ and for which*

$$\int_{\Omega} |f(x,y)|^p \, dx\, dy < \infty.$$

Then $\mathcal{F}$ is a normal family.

We now prove Theorem 2.1.

Proof of Theorem 2.1. For $t \in (0, T_1]$, let

$$\Omega(t) = \bigcup_{\alpha \in S_f:\, |\alpha| t^{1/2} \leq (2\, C_0)^{-1}} \{x + iy \in \mathbb{C}^3 :\ x \in 2B_*,\ \ y = \alpha\, \psi(x)\, t\},$$

and

$$Q_{T_1} = \left\{(x + iy, t) \in \mathbb{C}^3 \times (0, T_1] : x + iy \in \Omega(t)\right\}.$$

For each $\alpha \in S_f$ the contraction mapping principle guarantees that $U_\alpha^{(n)} + i V_\alpha^{(n)}$ and $P_\alpha^{(n)} + i \Pi_\alpha^{(n)}$ converge to functions $U_\alpha + i V_\alpha$ and $P_\alpha + i \Pi_\alpha$ where $U_\alpha, V_\alpha, P_\alpha, \Pi_\alpha \in C((0, T_1), L^q(\mathbb{R}^3))$ and solve the appropriate limiting systems.

Note that by Lemma 3.2 and for all $t \in (0, T_1]$ and $n \in \mathbb{N}$ we have the uniform bound

$$\int_{\Omega(t)} \left|(U^{(n)} + i V^{(n)})(x,y)\right|^q dx\, dy \leq \left(\frac{2\, T_1^{1/2}}{C_0}\right)^3 M_0.$$

For any $t \in (0, T_1]$, applying Lemma 3.3 shows that $\{U^{(n)} + i V^{(n)}\}_{n \in \mathbb{N}}$ belongs to a normal family and so there exists a subsequence $\{U^{(n_k)} + i V^{(n_k)}\}_{k \in \mathbb{N}}$ which converges to an analytic function on compact subsets of $\Omega(t)$. We can thus undo the change of variables $y = \alpha\, t\, \psi(x)$ using the Cauchy–Riemann system to obtain $U + iV$ from the functions $U_\alpha + i V_\alpha$ and conclude that $U + iV$ solves the complexified Navier–Stokes equations on Q_{T_1}. More precisely, consider $(x_0 + i\alpha\psi(x_0)t, t) \in Q_{T_1}$ and let $K \subset \Omega(t)$ be a compact set containing $x_0 + i\alpha\psi(x_0)t$. Then, there exists a function $U + iV$ which is analytic at $x_0 + i\alpha\psi(x_0)t$ and the uniform limit of $U^{(n_k)} + i V^{(n_k)}$ on K for some subsequence $\{n_k\}$. There thus exists a neighborhood of x_0 so that, since $U^{(n_k)}(x, \alpha\psi(x)t, t) = U_\alpha^{(n_k)}(x, t)$, we have $U(x, \alpha\psi(x)t, t) = U_\alpha(x, t)$. Since $U + iV$ is analytic in a neighborhood of $x_0 + i\alpha\psi(x_0)t$, we apply the Cauchy–Riemann system to the limiting systems solved by U_α and V_α and conclude that $U + iV$ satisfies the complexified Navier–Stokes equations at $(x_0 + i\alpha\psi(x_0)t, t)$. The definition of $U + iV$ can be uniquely extended to all interior points of $\Omega(t)$ in such a way that it's components are the limit of $U^{(n)}$ and $V^{(n)}$ in L^q and $U + iV$ is an analytic solution to the complexified Navier–Stokes equations.

We finally mention why u is the restriction of $U + iV$ to $\mathbb{R}^3$. Setting $\alpha = 0$ we see that $U(x, 0, t) + iV(x, 0, t) = U(x, 0, t)$ is a mild solution to the 3D NSE with initial data in $L^q(\mathbb{R}^3)$ and, since $q > 3$, must agree with the unique mild solution u. □

4. Locally analytic forcing and purely local dependence

In this section we sketch the proof of Theorem 2.3. Let ϕ denote a non-negative test function evaluating to 1 on $2B_*$ and supported on $4B_*$. Assuming u solves the 3D NSE and is smooth on $4B_* \times (0, T)$ we derive an evolution equation

$$\partial_t(\phi u) - \Delta(\phi u) = -(\phi u) \cdot \nabla(\phi u) - \nabla(\phi p) + \Phi_1 + \phi f,$$

where

$$\Phi_1 = \phi(1 - \phi)\nabla \cdot (u \otimes u) + (\nabla\phi \cdot u)\phi u - 2\nabla\phi \cdot \nabla u - 2\Delta\phi\, u + p\nabla\phi.$$

Also, ϕp satisfies

$$\Delta(\phi p) = -\partial_i\partial_j(\phi u_i \phi u_j) + \Phi_2,$$

where

$$\Phi_2 = (1 - \phi)\,\phi\,\partial_i\partial_j(u_i u_j) + \partial_i\partial_j(\phi^2)u_i u_j + 2\nabla\phi\nabla p + (\Delta\phi)\,p.$$

Let $\mu_0 = \phi u_0$ and construct a sequence of functions $\mu^{(n)}$ and $\rho^{(n)}$ by first letting $\mu^{(0)} = \rho^{(0)} = 0$ and then iteratively solving

$$\begin{aligned}
&\partial_t\mu^{(n)} - \Delta\mu^{(n)} = -\mu^{(n-1)} \cdot \nabla\mu^{(n-1)} - \nabla\rho^{(n-1)} + \Phi_1 + \phi f && \text{in } \mathbb{R}^3 \times (0, T)\\
&\Delta\rho^{(n-1)} = -\partial_i\partial_j\big(\mu_i^{(n-1)}\mu_j^{(n-1)}\big) + \Phi_2 && \text{in } \mathbb{R}^3 \times (0, T)\\
&\mu^{(n)}(\cdot, 0) = \mu_0(\cdot) && \text{in } \mathbb{R}^3.
\end{aligned}$$

This scheme formally resembles the one given for $u^{(n)}$ and $p^{(n)}$ at the beginning of Section 3. It is possible to show that the limit of $\mu^{(n)}$ coincides with ϕu on $(0, T_2)$ where T_2 is as defined in Theorem 2.3. We sketch the details. Establishing the convergence of $\mu^{(n)}$ to some limiting field μ is done by a contractive argument. Then, considering $\widetilde{\mu} = \mu - \phi u$ and noting that both μ and ϕu are smooth, we use the mild solution formula for $\widetilde{\mu}(\cdot, t)$ and apply a standard argument (cf. [7]) to conclude $\widetilde{\mu}(\cdot, t) = 0$ for $t \in (0, T_3)$ provided T_3 is sufficiently small. This process is iterated to obtain the conclusion on all of $(0, T_2)$.

Proof of Theorem 2.3. Since Φ_1 and Φ_2 are zero on $2B_*$ and, noting the analyticity properties of f, the solutions $\mu^{(n)}$ and $\rho^{(n)}$ agree with the restrictions to $2B_*$ of analytic functions which we denote by $U^{(n)} + iV^{(n)}$ and $P^{(n)} + i\Pi^{(n)}$. As above, we use the subscript α to denote evaluation at $y = \alpha\,\psi(x)\,t$ where $\alpha \in S_f$, the support of ψ is $2B_*$, and $t \in (0, T)$. These functions satisfy systems which are similar to (3.4)–(3.7), the only difference being the presence of Φ_1 and Φ_2 in the non-homogeneous parts of the equations for $U^{(n)}$ and $P^{(n)}$.

Estimates in $L^\infty(0, T_2; L^q(\mathbb{R}^3))$ for some $T_2 > 0$ are established using an inductive argument similar to that in Section 3. Here, however, we need to control the terms from the integral equations for $U_\alpha^{(n)}$ and $\Pi_\alpha^{(n)}$ involving Φ_1 and Φ_2. For Φ_1 it is straightforward to show that whenever $0 < t < T$ we have

$$
\begin{aligned}
&\Big\| \int_0^t e^{(t-\tau)\Delta}\Phi_1(\tau)\,d\tau \Big\|_{L^q(\mathbb{R}^3)} \\
&\le C\,T^{(q-3)/(2q)} \big(\sup_{0<\tau<T} ||u(\tau)||^2_{L^q(4B_*)} + \sup_{0<\tau<T} ||p(\tau)||_{L^{q/2}(4B_*)} \big) \\
&\quad + C\,T^{(r-1)/(r)-3/(2q)} \sup_{0<\tau<T} ||u(\tau)||_{L^q(4B_*)} \Big(\int_0^t ||\nabla u(\tau)||^r_{L^q(4B_*)}\,d\tau \Big)^{\frac{1}{r}} \\
&\quad + C\,T^{(r-2)/(2r)} \Big(\int_0^t ||\nabla u(\tau)||^r_{L^q(4B_*)}\,d\tau \Big)^{\frac{1}{r}} + C \sup_{0<\tau<T} ||u(\tau)||_{L^q(4B_*)},
\end{aligned}
\tag{4.1}
$$

provided $T < r_*^2$. At the level of gradients we have the L^r–L^q estimate

$$
\begin{aligned}
&\Big(\int_0^T \Big\| \int_0^t \nabla e^{(t-\tau)\Delta}\Phi_1(\tau)\,d\tau \Big\|^r_{L^q(\mathbb{R}^3)} dt \Big)^{\frac{1}{r}} \\
&\le C\,T^{(q-3)/(2q)} \sup_{0<\tau<t} ||u(\tau)||_{L^q(4B_*)} \Big(\int_0^T \|\nabla u(\tau)\|^r_{L^q(4B_*)}\,d\tau \Big)^{\frac{1}{r}} \\
&\quad + C\,T^{1/r-3/(2q)} \Big(\sup_{0<\tau<T} ||u(\tau)||^2_{L^q(4B_*)} + \sup_{0<\tau<T} ||p(\tau)||_{L^{q/2}(4B_*)} \Big) \\
&\quad + C\,T^{1/r-1/2} \sup_{0<\tau<T} ||u(\tau)||_{L^q(4B_*)}.
\end{aligned}
\tag{4.2}
$$

For terms involving Φ_2 note that $P_\alpha^{(n)}$ satisfies an integral formula comprised of the terms I and J defined in Section 3 with an additional term involving Φ_2, i.e.,

$$P_\alpha^{(n)}(x) = I + J + K,$$

where

$$K(x,t) = C \int_{\mathbb{R}^3} \frac{1}{|x-y|}\Phi_2(y,t)\,dy.$$

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The $L^{q/2}$-norm of ∇K satisfies

$$\|\nabla K\|_{L^{q/2}(\mathbb{R}^3)} \le C\,\|u\|_{L^q(4B_*)}\|\nabla u\|_{L^q(4B_*)} + C\,r_*^{-1}||p||_{L^{q/2}(4B_*)}, \tag{4.3}$$

which can be verified using the Calderon–Zygmund and the Hardy–Littlewood–Sobolev inequalities. This leads to the estimates

$$\begin{aligned}
&\left\| \int_0^t e^{(t-\tau)\Delta} b_{lk}^{11} \partial_k P_\alpha^{(n)}\, d\tau \right\|_{L^q(\mathbb{R}^3)} \\
&\le C\,T^{(r-1)/r-3/(2q)} \big(\sup_{0<\tau<T} ||U_\alpha^{(n)}(\tau)||_{L^q(\mathbb{R}^3)} + \sup_{0<\tau<T} ||V_\alpha^{(n)}(\tau)||_{L^q(\mathbb{R}^3)} \big) \\
&\quad \times \left(\left(\int_0^T \|\nabla U_\alpha^{(n)}(\tau)\|_{L^q(\mathbb{R}^3)}^r\, d\tau \right)^{\frac{1}{r}} + \left(\int_0^T \|\nabla V_\alpha^{(n)}(\tau)\|_{L^q(\mathbb{R}^3)}^r\, d\tau \right)^{\frac{1}{r}} \right) \\
&\quad + C\,T^{(r-1)/r-3/(2q)} \sup_{0<t<T} \|u\|_{L^q(4B_*)} \left(\int_0^T \|\nabla u(t)\|_{L^q(4B_*)}^r\, dt \right)^{\frac{1}{r}} \\
&\quad + C\,T^{(q-3)/(2q)} \sup_{0<t<T} \|p\|_{L^{q/2}(4B_*)},
\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}
&\left(\int_0^T \left\| \int_0^t \nabla e^{(t-\tau)\Delta} b_{lk}^{11} \partial_k P_\alpha^{(n)}(\tau)\, d\tau \right\|_{L^q(\mathbb{R}^3)}^r dt \right)^{1/r} \\
&\le C\,T^{(q-3)/(2q)} \big(\sup_{0<\tau<T} ||U_\alpha^{(n)}(\tau)||_{L^q(\mathbb{R}^3)} + \sup_{0<\tau<T} ||V_\alpha^{(n)}(\tau)||_{L^q(\mathbb{R}^3)} \big) \\
&\quad \times \left(\left(\int_0^T \|\nabla U_\alpha^{(n)}(\tau)\|_{L^q(\mathbb{R}^3)}^r\, d\tau \right)^{\frac{1}{r}} + \left(\int_0^T \|\nabla V_\alpha^{(n)}(\tau)\|_{L^q(\mathbb{R}^3)}^r\, d\tau \right)^{\frac{1}{r}} \right) \\
&\quad + C\,T^{(q-3)/(2q)} \sup_{0<t<T} \|u\|_{L^q(4B_*)} \left(\int_0^T \|\nabla u(t)\|_{L^q(4B_*)}^r\, dt \right)^{\frac{1}{r}} \\
&\quad + C\,T^{1/r-3/(2q)} \sup_{0<t<T} \|p\|_{L^{q/2}(4B_*)}.
\end{aligned} \tag{4.5}$$

Let $\widetilde{U}_\alpha^{(n)} = U_\alpha^{(n)} - \phi\, u$. Using (4.1), (4.4), and estimates analogous to those contained in Section 3, we obtain a bound similar to (3.17), namely

$$\begin{aligned}
&\|\widetilde{U}_\alpha^{(n+1)}(t)\|_{L^q(\mathbb{R}^3)} + \|V_\alpha^{(n+1)}(t)\|_{L^q(\mathbb{R}^3)} \\
&\quad \le T\,M_{f,\mathrm{loc}} + (C|\alpha|T^{1/2} + 1)\,M_{\mathrm{loc}} + T^{(q-3)/(2q)}\,M_{\mathrm{loc}}^2
\end{aligned}$$

$$+ C\,|\alpha|\,T^{(r-1)/r}\Big(\Big(\int_0^T \|\nabla \widetilde{U}_\alpha^{(n+1)}(t)\|^r_{L^q(\mathbb{R}^3)}\,dt\Big)^{\frac{1}{r}} + \Big(\int_0^T \|\nabla V_\alpha^{(n+1)}(t)\|^r_{L^q(\mathbb{R}^3)}\,dt\Big)^{\frac{1}{r}}\Big)$$

$$+ C\,T^{(r-1)/r-3/(2q)}\big(\sup_{0<\tau<T}||U_\alpha^{(n)}(\tau)||_{L^q(\mathbb{R}^3)} + \sup_{0<\tau<T}||V_\alpha^{(n)}(\tau)||_{L^q(\mathbb{R}^3)}\big)$$

$$\times\Big(\Big(\int_0^T \|\nabla U_\alpha^{(n)}(\tau)\|^r_{L^q(\mathbb{R}^3)}\,d\tau\Big)^{\frac{1}{r}} + \Big(\int_0^T \|\nabla V_\alpha^{(n)}(\tau)\|^r_{L^q(\mathbb{R}^3)}\,d\tau\Big)^{\frac{1}{r}}\Big). \tag{4.6}$$

Also, using (4.2), (4.5), and the approach of Section 3, we obtain a bound similar to (3.22), namely

$$\Big(\int_0^T \|\nabla \widetilde{U}_\alpha^{(n+1)}(t)\|^r_{L^q(\mathbb{R}^3)}\,dt\Big)^{\frac{1}{r}} + \Big(\int_0^T \|\nabla V_\alpha^{(n+1)}(t)\|^r_{L^q(\mathbb{R}^3)}\,dt\Big)^{\frac{1}{r}}$$

$$\le T^{(2+r)/(2r)}M_{f,\mathrm{loc}} + C\,|\alpha|\,T^{1/r}\,M_{\mathrm{loc}} + C\,T^{(q-3)/(2q)-(r-2)/(2r)}\,M_{\mathrm{loc}}^2$$

$$+ C\,|\alpha|\,T^{1/r}\Big(\sup_{0<\tau<T}\|\widetilde{U}_\alpha^{(n+1)}(\tau)\|_{L^q(\mathbb{R}^3)} + \sup_{0<\tau<T}\|V_\alpha^{(n+1)}(\tau)\|_{L^q(\mathbb{R}^3)}\Big)$$

$$+ C\,|\alpha|\,T^{1/2}\Big(\Big(\int_0^T \|\nabla \widetilde{U}_\alpha^{(n+1)}(t)\|^r_{L^q(\mathbb{R}^3)}\,dt\Big)^{\frac{1}{r}} + \Big(\int_0^T \|\nabla V_\alpha^{(n+1)}(t)\|^r_{L^q(\mathbb{R}^3)}\,dt\Big)^{\frac{1}{r}}\Big)$$

$$+ C\,T^{(q-3)/(2q)}\big(\sup_{0<\tau<T}||U_\alpha^{(n)}(\tau)||_{L^q(\mathbb{R}^3)} + \sup_{0<\tau<T}||V_\alpha^{(n)}(\tau)||_{L^q(\mathbb{R}^3)}\big)$$

$$\times\Big(\Big(\int_0^T \|\nabla U_\alpha^{(n)}(\tau)\|^r_{L^q(\mathbb{R}^3)}\,d\tau\Big)^{\frac{1}{r}} + \Big(\int_0^T \|\nabla V_\alpha^{(n)}(\tau)\|^r_{L^q(\mathbb{R}^3)}\,d\tau\Big)^{\frac{1}{r}}\Big). \tag{4.7}$$

Recalling the inductive argument from Section 3, it is clear that if

$$T_2 \le \frac{1}{C}\min\Big\{T,\ r_*^2,\ \Big(\frac{1}{M_{\mathrm{loc}}}\Big)^{2q/(q-3)},\ \frac{M_{\mathrm{loc}}}{M_{f,\mathrm{loc}}}\Big\}, \tag{4.8}$$

and if the local versions of the inductive hypotheses (3.11) and (3.12) hold for $t \in (0, T)$ with M_0 replaced with M_{loc}, then, for all $n \in \mathbb{N}$,

$$\sup_{0<t<T}\|U_\alpha^{(n)}(t)\|_{L^q(4B_*)} + \sup_{0<t<T}\|V_\alpha^{(n)}(t)\|_{L^q(4B_*)} \le 5\,M_{\mathrm{loc}}. \tag{4.9}$$

From here the proof of Theorem 2.3 proceeds identically to that of Theorem 2.1. □

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