

A Spatially Localized $L \log L$ Estimate on the Vorticity in the 3D NSE

ZACHARY BRADSHAW & ZORAN GRUJIĆ

ABSTRACT. The purpose of this note is to present a spatially localized $L \log L$ bound on the vorticity in the 3D Navier-Stokes equations, assuming a very mild, *purely geometric* condition. This yields an extra-log decay of the distribution function of the vorticity, which in turn implies *breaking the criticality* in a physically, numerically, and mathematical analysis-motivated criticality scenario based on vortex stretching and anisotropic diffusion.

1. INTRODUCTION

Three-dimensional Navier-Stokes equations (3D NSE)—describing a flow of 3D incompressible viscous fluid—read

$$u_t + (u \cdot \nabla)u = -\nabla p + \Delta u,$$

supplemented with the incompressibility condition $\operatorname{div} u = 0$, where u is the velocity of the fluid, and p is the pressure (here, the viscosity is set to 1). Applying the curl operator yields the vorticity formulation

$$(1.1) \quad \omega_t + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u + \Delta \omega,$$

where $\omega = \operatorname{curl} u$ is the vorticity of the fluid. If we take the spatial domain to be the whole space, the velocity can be recovered from the vorticity via the Biot-Savart Law,

$$u(x) = c \int \nabla \frac{1}{|x - y|} \times \omega(y) \, dy;$$

this makes equation (1.1) a closed (non-local) system for the vorticity field alone.

An *a priori* L^1 -bound on the evolution of the vorticity in the 3D NSE was obtained by Constantin in [4]. This is a vorticity analogue of Leray's *a priori* L^2 -bound on the velocity, and both quantities scale in the same fashion. So far, there has been no *a priori* estimate on the weak solutions to the 3D NSE breaking this scaling.

The goal of this short article is to show that a very mild, *purely geometric* assumption yields a uniform-in-time $L \log L$ bound on the vorticity, effectively breaking the aforementioned scaling. More precisely, the assumption is a uniform-in-time boundedness of the localized *vorticity direction* in the weighted bmo-space $\widetilde{\text{bmo}}_{1/|\log r|}$. An interesting feature of this space (cf. [11]) is that it allows for discontinuous functions exhibiting singularities of, for example,

$$\sin \log |\log(\text{something algebraic})| \text{-type.}$$

In particular, the vorticity direction can blow up in a geometrically spectacular fashion—every point on the unit sphere being a limit point—and the $L \log L$ bound will still hold.

Besides being of independent interest, the $L \log L$ bound on the vorticity implies an extra-log decay of the distribution function. This is significant, as it transforms a recently exposed [5, 9] large-data criticality scenario for the 3D NSE into a no blow-up scenario. Shortly, creation and persistence (in the sense of the time-average) of the *axial* filamentary scales comparable to the *macro-scale*, paired with the L^1 -induced decay of the volume of the suitably defined region of intense vorticity, leads to creation and persistence of the *transversal micro-scales* comparable to the scale of local, *anisotropic linear sparseness*, enabling the anisotropic diffusion to equalize the nonlinear effects. The extra log-decay of the volume transforms the equalizing scenario into the anisotropic diffusion-win scenario.

The present result is, in a way, complementary to the results obtained by the authors in [1]. The class of conditions leading to an $L \log L$ -bound presented in [1] can be characterized as “wild in time” with a uniform spatial (e.g., algebraic) structure, while the condition presented here can be characterized as “wild in space” and uniform in time. As in [1], the proof is based on an adaptation of the method exposed in [4], the novel component being utilization of *analytic cancelations* in the vortex-stretching term via the Hardy space-version of the Div-Curl Lemma [2], the local version [8] of \mathcal{H}^1 -BMO duality [6, 7], and the intimate connection between the BMO-norm and the logarithm. While the argument in [1] relied on the structure of the evolution of the scalar components and the result (cf. [14, 16]) stating that the BMO-norm of the logarithm of a polynomial is bounded *independently of the coefficients*, the present argument relies on sharp pointwise multiplier theorems in $\widetilde{\text{bmo}}$ [11–13] and Coifman-Rochberg's BMO-estimate on the logarithm of the *maximal function* of a locally integrable function [3], the estimate being *fully independent of the function* and depending only on the dimension of the space.

2. AN EXCURSION TO HARMONIC ANALYSIS

In this section, we compile several results from harmonic analysis that will prove useful.

Hardy spaces \mathcal{H}^1 and \mathfrak{h}^1 . The maximal function of a distribution f is defined as

$$M_h f(x) = \sup_{t>0} |f * h_t(x)|, \quad x \in \mathbb{R}^n,$$

where h is a fixed, normalized ($\int h \, dx = 1$) test function supported in the unit ball, and h_t denotes $t^{-n}h(\cdot/t)$.

The distribution f is in the Hardy space \mathcal{H}^1 if $\|f\|_{\mathcal{H}^1} = \|M_h f\|_1 < \infty$.

The local maximal function is defined as

$$m_h f(x) = \sup_{0<t<1} |f * h_t(x)|, \quad x \in \mathbb{R}^n,$$

and the distribution f is in the local Hardy space \mathfrak{h}^1 if $\|f\|_{\mathfrak{h}^1} = \|m_h f\|_1 < \infty$.

Div-Curl Lemma. (Coifman, Lions, Meyer, Semmes [2]) Suppose that E and B are L^2 -vector fields satisfying $\operatorname{div} E = \operatorname{curl} B = 0$ (in the sense of distribution). Then,

$$\|E \cdot B\|_{\mathcal{H}^1} \leq c(n) \|E\|_{L^2} \|B\|_{L^2}.$$

BMO and weighted bmo spaces. The classical space of bounded mean oscillations, BMO, is defined as

$$\text{BMO} = \{f \in L^1_{\text{loc}} \mid \sup_{x \in \mathbb{R}^3, r>0} \Omega(f, I(x, r)) < \infty\},$$

where $\Omega(f, I(x, r)) = (1/|I(x, r)|) \int_{I(x, r)} |f(x) - f_I| \, dx$ is the mean oscillation of the function f with respect to its mean $f_I = (1/|I(x, r)|) \int_{I(x, r)} f(x) \, dx$, over the cube $I(x, r)$ centered at x with the side-length r .

A local version of BMO, usually denoted by bmo , is defined by finiteness of the expression

$$\|f\|_{\text{bmo}} = \sup_{x \in \mathbb{R}^3, 0<r<\delta} \Omega(f, I(x, r)) + \sup_{x \in \mathbb{R}^3, r \geq \delta} \frac{1}{|I(x, r)|} \int_{I(x, r)} |f(y)| \, dy,$$

for some positive δ .

If $f \in L^1$, we can focus on small scales, for example, $0 < r < \frac{1}{2}$. Let φ be a positive, non-decreasing function on $(0, \frac{1}{2})$, and consider the following version of

local weighted spaces of bounded mean oscillations:

$$\|f\|_{\widetilde{\text{bmo}}_\varphi} = \|f\|_{L^1} + \sup_{x \in \mathbb{R}^3, 0 < r < 1/2} \frac{\Omega(f, I(x, r))}{\varphi(r)}$$

(cf. [12]).

Of special interest will be the spaces $\widetilde{\text{bmo}} = \widetilde{\text{bmo}}_1$ and $\widetilde{\text{bmo}}_{1/|\log r|}$.

\mathcal{H}^1 -BMO and \mathfrak{h}^1 -bmo duality (Fefferman [6, 7], Goldberg [8]). We have that $(\mathcal{H}^1)^* = \text{BMO}$ and $(\mathfrak{h}^1)^* = \text{bmo}$; the duality is realized via integration of one object against the other.

Pointwise multipliers in $\widetilde{\text{bmo}}$ ([11–13]). We have the following theorem.

Theorem (A sharp pointwise multiplier theorem [12]). *Let h be in $\widetilde{\text{bmo}}$, and g in $L^\infty \cap \widetilde{\text{bmo}}_{1/|\log r|}$. Then,*

$$\|gh\|_{\widetilde{\text{bmo}}} \leq c(n) (\|g\|_\infty + \|g\|_{\widetilde{\text{bmo}}_{1/|\log r|}}) \|h\|_{\widetilde{\text{bmo}}}.$$

More precisely, the space of pointwise $\widetilde{\text{bmo}}$ multipliers coincides with $L^\infty \cap \widetilde{\text{bmo}}_{1/|\log r|}$.

Coifman and Rochberg’s estimate on $\|\log Mf\|_{\text{BMO}}$ ([3]). We let M denote the Hardy-Littlewood maximal operator. Coifman and Rochberg [3] obtained a characterization of BMO in terms of images of the logarithm of the maximal function of non-negative locally integrable functions (plus a bounded part). The main ingredient in demonstrating one direction is the estimate

$$\|\log Mf\|_{\text{BMO}} \leq c(n),$$

for any locally integrable function f . (Note that the bound is completely *independent* of f .)

This estimate remains valid if we replace Mf with $\mathcal{M}f = (M\sqrt{|f|})^2$ (cf. [10]); the advantage of working with \mathcal{M} is that the L^2 -maximal theorem implies the estimate

$$(2.1) \quad \|\mathcal{M}f\|_1 \leq c(n) \|f\|_1,$$

a bound that does not hold for the original maximal operator M .

3. SETTING AND THE RESULT

Consider a weak (distributional) Leray solution u on \mathbb{R}^3 . The vorticity analogue of the Leray’s *a priori* bound on the energy was presented in [4]: assuming that the initial vorticity is in L^1 (or, more generally, a bounded measure), the L^1 -norm of the vorticity remains bounded on any finite time-interval.

Our goal here is to obtain a spatially localized $L \log L$ bound on the vorticity under a suitable assumption on the structural blow-up of the *vorticity direction* $\xi = \omega/|\omega|$.

We fix a spatial ball $B(0, R_0)$, and consider a test function ψ supported in $B = B(0, 2R_0)$ such that $\psi = 1$ on $B(0, R_0)$ and $|\nabla\psi(x)| \leq c(1/R_0)\psi^\delta(x)$ for some $\delta > 0$.

Let $w = \sqrt{1 + |\omega|^2}$. We aim to control the evolution of $\psi w \log w$; by Stein’s lemma [15], this is essentially equivalent to controlling the L^1 -norm of Mw .

For simplicity of the exposition, we assume that the initial vorticity is also in L^2 , and that $T > 0$ is the first (possible) blow-up time. This way, the solution in view is smooth on $(0, T)$, and we can focus on obtaining a $\sup_{t \in (0, T)}$ -bound. Alternatively, one can employ the retarded mollifiers.

Theorem 3.1. *Let u be a Leray solution to the 3D NSE. Assume that the initial vorticity ω_0 is in $L^1 \cap L^2$, and that $T > 0$ is the first (possible) blow-up time. Suppose that*

$$\sup_{t \in (0, T)} \|(\psi\xi)(\cdot, t)\|_{\widetilde{bmo}_{1/|\log r|}} < \infty.$$

Then,

$$\sup_{t \in (0, T)} \int \psi(x)w(x, t) \log w(x, t) \, dx < \infty.$$

Remark 3.2. Since ω_0 is in L^1 , in addition to the Leray’s *a priori* bounds on u , namely,

$$\sup_t \|u(\cdot, t)\|_{L^2} < \infty \quad \text{and} \quad \int_t \int_x |\nabla u(x, t)|^2 \, dx \, dt < \infty,$$

we have that the following *a priori* bounds on ω are also at our disposal [4]:

$$\sup_t \|\omega(\cdot, t)\|_{L^1} < \infty \quad \text{and} \quad \int_t \int_x |\nabla \omega(x, t)|^{4/(3+\varepsilon)} \, dx \, dt < \infty.$$

Proof. Setting $q(y) = \sqrt{1 + |y|^2}$, the evolution of $w = \sqrt{1 + |\omega|^2}$ satisfies the following partial differential inequality ([4]):

$$(3.1) \quad \partial_t w - \Delta w + (u \cdot \nabla)w \leq \omega \cdot \nabla u \cdot \frac{\omega}{w}.$$

Since our goal is to control the evolution of $\psi w \log w$, it will prove convenient to multiply (3.1) by $\psi(1 + \log w)$. Below is the calculus corresponding to each of the four terms.

Time-derivative:

$$\partial_t w \times \psi(1 + \log w) = \partial_t(\psi w \log w).$$

Laplacian:

$$\begin{aligned} -\Delta w \times \psi(1 + \log w) &= -\Delta(\psi w \log w) + \Delta \psi w \log w \\ &\quad + \psi \frac{1}{w} \sum_i (\partial_i w)^2 + 2 \sum_i \partial_i \psi \partial_i w (1 + \log w). \end{aligned}$$

Advection:

$$\begin{aligned} &(u \cdot \nabla) w \times \psi(1 + \log w) \\ &= \sum_i u_i \partial_i w \psi(1 + \log w) \\ &= \sum_i (\partial_i(u_i w \psi(1 + \log w)) - u_i w \partial_i \psi(1 + \log w) - u_i \psi \partial_i w) \\ &= \sum_i (\partial_i(u_i w \psi(1 + \log w)) - u_i w \partial_i \psi(1 + \log w) - \partial_i(u_i \psi w) + (u_i \partial_i \psi w)). \end{aligned}$$

Vortex-stretching:

$$\begin{aligned} \omega \cdot \nabla u \cdot \frac{\omega}{w} \times \psi(1 + \log w) &= \omega \cdot \nabla u \cdot \psi \frac{\omega}{|\omega|} (1 + \log w) \\ &\quad + \omega \cdot \nabla u \cdot \psi \left(\frac{\omega}{w} - \frac{\omega}{|\omega|} \right) (1 + \log w). \end{aligned}$$

Integrating over the space-time, the above representation yields (dropping the zero and the positive terms, and estimating the remaining terms in the straightforward fashion via Hölder and Sobolev) the following:

$$\begin{aligned} I(\tau) &\equiv \int \psi(x) w(x, \tau) \log w(x, \tau) \, dx \\ &\leq I(0) + c \int_0^\tau \int_x \omega \cdot \nabla u \cdot \psi \xi \log w \, dx \, dt + a \text{ priori} \end{aligned}$$

bounded, for any τ in $[0, T)$.

In order to take the advantage of the Coifman-Rochberg estimate, we decompose the logarithmic factor as

$$\log w = \log \frac{w}{\mathcal{M}w} + \log \mathcal{M}w.$$

Denoting $\int_0^\tau \int_x \omega \cdot \nabla u \cdot \psi \xi \log w \, dx \, dt$ by J , we obtain $J = J_1 + J_2$, where

$$J_1 = \int_0^\tau \int_x \omega \cdot \nabla u \cdot \psi \xi \log \frac{w}{\mathcal{M}w} \, dx \, dt$$

and

$$J_2 = \int_0^\tau \int_x \omega \cdot \nabla u \cdot \psi \xi \log \mathcal{M}w \, dx \, dt.$$

For J_1 , we use the *pointwise* inequality

$$w \log \frac{w}{\mathcal{M}w} \leq \mathcal{M}w - w$$

(a consequence of the pointwise inequality $\mathcal{M}f \geq f$, and the inequality $e^{x-1} \geq x$ for $x \geq 1$).

This leads to

$$J_1 \leq \int_0^\tau \int_x |\nabla u| (\mathcal{M}w - w) \psi \, dx \, dt,$$

which is *a priori* bounded by the Cauchy-Schwarz and the L^2 -maximal theorem.

For J_2 , we have the following string of inequalities:

$$\begin{aligned} J_2 &\leq c \int_0^\tau \|\omega \cdot \nabla u\|_{\mathfrak{h}^1} \|\psi \xi \log \mathcal{M}w\|_{\text{bmo}} \, dt \\ &\leq c \int_0^\tau \|\omega \cdot \nabla u\|_{\mathfrak{H}^1} \|\psi \xi \log \mathcal{M}w\|_{\widetilde{\text{bmo}}} \, dt \\ &\leq c \int_0^\tau \|\omega\|_2 \|\nabla u\|_2 \left(\|\psi \xi\|_\infty + \|\psi \xi\|_{\widetilde{\text{bmo}}_{1/|\log r|}} \right) \\ &\quad \times \left(\|\log \mathcal{M}w\|_{\text{BMO}} + \|\log \mathcal{M}w\|_1 \right) \, dt \\ &\leq c \sup_{t \in (0, T)} \left\{ \left(1 + \|\psi \xi\|_{\widetilde{\text{bmo}}_{1/|\log r|}} \right) \left(\|\log \mathcal{M}w\|_{\text{BMO}} + \|\log \mathcal{M}w\|_1 \right) \right\} \\ &\quad \times \int_t \int_x |\nabla u|^2 \, dx \, dt \\ &\leq c \left(1 + \sup_{t \in (0, T)} \|\omega\|_1 \right) \left(1 + \sup_{t \in (0, T)} \|\psi \xi\|_{\widetilde{\text{bmo}}_{1/|\log r|}} \right) \int_t \int_x |\nabla u|^2 \, dx \, dt \end{aligned}$$

by \mathfrak{h}^1 -bmo duality, the Div-Curl Lemma, the pointwise $\widetilde{\text{bmo}}$ -multiplier theorem, the Coifman-Rochberg estimate, and the bound (2.1) combined with a couple of elementary inequalities. This completes the proof of the $L \log L$ estimate. \square

Acknowledgements. Zachary Bradswaw acknowledges the support of the *Virginia Space Grant Consortium* via the Graduate Research Fellowship. Zoran Grujić acknowledges the support of the *Research Council of Norway* via the grant 213474/F20, and the *National Science Foundation* via the grant DMS 1212023. We are grateful to Luong Dang Ky for the comments on the previous version of the paper.

REFERENCES

- [1] Z. BRADSHAW and Z. GRUJIĆ, *Blow-up scenarios for the 3D Navier-Stokes equations exhibiting sub-criticality with respect to the scaling of one-dimensional local sparseness*, J. Math. Fluid Mech. **16** (2014), no. 2, 321–334. <http://dx.doi.org/10.1007/s00021-013-0155-0>. MR3208718
- [2] R. R. COIFMAN, P. L. LIONS, Y. MEYER, and S. SEMMES, *Compensated compactness and Hardy spaces*, J. Math. Pures Appl. (9) **72** (1993), no. 3, 247–286 (English, with English and French summaries). MR1225511 (95d:46033)
- [3] R. R. COIFMAN and R. ROCHBERG, *Another characterization of BMO*, Proc. Amer. Math. Soc. **79** (1980), no. 2, 249–254. <http://dx.doi.org/10.2307/2043245>. MR565349 (81b:42067)
- [4] P. CONSTANTIN, *Navier-Stokes equations and area of interfaces*, Comm. Math. Phys. **129** (1990), no. 2, 241–266. <http://dx.doi.org/10.1007/BF02096982>. MR1048693 (91i:35148)
- [5] R. DASCALIUC and Z. GRUJIĆ, *Vortex stretching and criticality for the three-dimensional Navier-Stokes equations*, J. Math. Phys. **53** (2012), no. 11, 115613, 9. <http://dx.doi.org/10.1063/1.4752170>. MR3026558
- [6] Ch. FEFFERMAN, *Characterizations of bounded mean oscillation*, Bull. Amer. Math. Soc. **77** (1971), no. 4, 587–588. <http://dx.doi.org/10.1090/S0002-9904-1971-12763-5>. MR0280994 (43 #6713)
- [7] Ch. FEFFERMAN and E. M. STEIN, *H^p spaces of several variables*, Acta Math. **129** (1972), no. 1, 137–193. <http://dx.doi.org/10.1007/BF02392215>. MR0447953 (56 #6263)
- [8] D. GOLDBERG, *A local version of real Hardy spaces*, Duke Math. J. **46** (1979), no. 1, 27–42. <http://dx.doi.org/10.1215/S0012-7094-79-04603-9>. MR523600 (80h:46052)
- [9] Z. GRUJIĆ, *A geometric measure-type regularity criterion for solutions to the 3D Navier-Stokes equations*, Nonlinearity **26** (2013), no. 1, 289–296. <http://dx.doi.org/10.1088/0951-7715/26/1/289>. MR3001772
- [10] T. IWANIEC and A. VERDE, *On the operator $\mathcal{L}(f) = f \log |f|$* , J. Funct. Anal. **169** (1999), no. 2, 391–420. <http://dx.doi.org/10.1006/jfan.1999.3443>. MR1730563 (2001f:42034)
- [11] S. JANSON, *On functions with conditions on the mean oscillation*, Ark. Mat. **14** (1976), no. 2, 189–196. <http://dx.doi.org/10.1007/BF02385834>. MR0438030 (55 #10951)
- [12] V. G. MAZ'YA and T. O. SHAPOSHNIKOVA, *Theory of Multipliers in Spaces of Differentiable Functions*, Monographs and Studies in Mathematics, vol. 23, Pitman (Advanced Publishing Program), Boston, MA, 1985. MR785568 (87j:46074)
- [13] E. NAKAI and K. YABUTA, *Pointwise multipliers for functions of bounded mean oscillation*, J. Math. Soc. Japan **37** (1985), no. 2, 207–218. <http://dx.doi.org/10.2969/jmsj/03720207>. MR780660 (87d:42020)
- [14] F. RICCI and E. M. STEIN, *Harmonic analysis on nilpotent groups and singular integrals. I: Oscillatory integrals*, J. Funct. Anal. **73** (1987), no. 1, 179–194. [http://dx.doi.org/10.1016/0022-1236\(87\)90064-4](http://dx.doi.org/10.1016/0022-1236(87)90064-4). MR890662 (88g:42023)
- [15] E. M. STEIN, *Note on the class $L \log L$* , Studia Math. **32** (1969), 305–310. MR0247534 (40 #799)
- [16] E. M. Stein (ed.), *Beijing Lectures in Harmonic Analysis*, Annals of Mathematics Studies, vol. 112, Princeton University Press, Princeton, NJ, 1986. Proceedings of a summer symposium on analysis in China held at Beijing University, Beijing, September 1984. MR864369 (87g:42001)

Department of Mathematics
 University of Virginia
 Kerchof Hall
 Charlottesville, VA 22904, USA
 E-MAIL, Zoran Grujić: zg7c@virginia.edu
 E-MAIL, Zachary Bradshaw: zb8br@virginia.edu

KEY WORDS AND PHRASES: 3D Navier-Stokes, criticality, vorticity.

2010 MATHEMATICS SUBJECT CLASSIFICATION: 35, 76.

Received: October 2, 2013.