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The Derivative Nonlinear Schrödinger Equation in Analytic Classes

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Abstract

The derivative nonlinear Schrödinger equation is shown to be locally well-posed in a class of functions analytic on a strip around the real axis. The main feature of the result is that the width of the strip *does not shrink* in time. To overcome the derivative loss, Kato-type smoothing results and space-time estimates are used.

1 Introduction

The aim of this article is to prove local well-posedness of the Cauchy problem for the derivative nonlinear Schrödinger equation

$$\begin{aligned} &iu_t + u_{xx} = i\left(|u|^2 u\right)_x, \\ &u(x,0) = u_0(x), \end{aligned} \tag{1.1}$$

with initial data u_0 whose real and imaginary parts are real-analytic functions with the uniform radii of analyticity σ.

The problem has been studied by many authors, among them Hayashi and Ozawa [13, 14, 15] and Kato and Masuda [16]. Hayashi and Ozawa proved local well-posedness in a class of analytic functions similar to the class considered in the present article. However, in their result, it was essential that the width of the strip of analyticity was allowed to shrink as the solution progressed in time. Our goal is to show that it is possible to obtain local well-posedness *without shrinking* the width of the strip (or, equivalently, without decreasing the uniform radius of spatial analyticity) in time. One difficulty in treating the equation is the so-called derivative loss. In [2], Bourgain introduced special function spaces which are well-suited to overcome this problem which also occurs in the Korteweg-de Vries equation. His method was refined by Kenig, Ponce and Vega and has been used by many other authors (cf. [3, 4, 5, 7, 17, 18]). In the context of the derivative Schrödinger equations, Takaoka has recently given a proof of the well-posedness of the equation in low-regularity classes [22]. Our proof relies on some of his estimates.

The function spaces considered in this article are known as analytic Gevrey spaces, and can be defined as follows (cf. [6]). For $\sigma > 0$ and $s \in \mathbb{R}$, define $G^{\sigma,s}$ to be the subspace of $L^2(\mathbb{R})$ for which

$$\|u_0\|^2_{G^{\sigma,s}} = \int_{-\infty}^{\infty} (1+|\xi|)^{2s} e^{2\sigma(1+|\xi|)} |\hat{u}_0(\xi)|^2 \, d\xi$$

is finite. The main result of this article is the following theorem.

Theorem 1. *Let $s > \frac{1}{2}$ and $\sigma > 0$. For initial data in $G^{\sigma,s}$, there exists a positive time T such that the initial-value problem (1.1) is well-posed in the space $C([0,T], G^{\sigma,s})$.*

Well-posedness includes existence of a solution, uniqueness, and continuous dependence on the initial data. Moreover, it is required that the solution describe a continuous curve in the solution space. This requirement is equivalent to membership in the class $C([0,T], G^{\sigma,s})$ of continuous functions mapping the time interval $[0,T]$ into $G^{\sigma,s}$.

Since the nonlinear term $i\partial_x(|u|^2 u)$ is difficult to treat, it is necessary to first apply a gauge transformation, as it has been done in many previous works (cf. [9, 13]). Suppose $u(x,t)$ is a solution of (1.1), and define a function $v(x,t)$ by

$$v(x,t) = e^{-i\int_{-\infty}^{x} |u(y,t)|^2 \, dy} \, u(x,t).$$

Then v satisfies the initial-value problem

$$\begin{aligned} & iv_t + v_{xx} = -iv^2 \bar{v}_x - \frac{1}{2}|v|^4 v, \\ & v(x,0) = v_0(x), \end{aligned} \tag{1.2}$$

with initial data defined by $v_0(x) = e^{-i\int_{-\infty}^{x} |u_0(y)|^2 \, dy} \, u_0(x)$. To see that this transform is continuous in the $G^{\sigma,s}$-norm, first consider the case when $s = 0$. Then one may use the equivalent norm

$$\|v_0\|^2_{\tilde{G}^{\sigma,0}} = \int_{-\infty}^{\infty} |v_0(x+i\sigma)|^2 \, dx + \int_{-\infty}^{\infty} |v_0(x-i\sigma)|^2 \, dx.$$

By complexifying the path integral in the definition of v, it is easily seen that the gauge transformation is continuous with respect to this norm. When s is an integer, one may differentiate v with respect to x to see that the gauge transform is continuous. Finally, the general case can be obtained by interpolation. The gauge transformation results in an equation which still has a derivative nonlinearity. However, the derivative nonlinearity appearing in (1.2) can be controlled using the space-time norms. The quintic nonlinearity does not pose any special challenge.

2 Auxiliary estimates

The Fourier transform of a function v_0 belonging to the Schwartz class is defined by

$$\hat{v}_0(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v_0(x) e^{-ix\xi} \, dx.$$

Defining a Fourier multiplier operator A by

$$\widehat{Av_0}(\xi) = (1 + |\xi|)\hat{v}_0(\xi),$$

the Gevrey norm of order (σ, s) can be written as

$$\|v_0\|_{G^{\sigma,s}} = \|A^s e^{\sigma A} v_0\|_{L^2(\mathbb{R})}.$$

In case a function under consideration is complex-valued (e.g., a solution of a Schrödinger equation), its real and imaginary parts are real-analytic with the uniform radii of convergence equal to σ. To prove local well-posedness, another family of function spaces is needed. For a function $v(x,t)$ of two variables, the notation $\hat{v}(\xi, \tau)$ is used to denote the space-time Fourier transform. Given $\sigma > 0$, $s \in \mathbb{R}$, and $b \in [-1, 1]$, define $X_{\sigma,s,b}$ to be the Banach space with the space-time norm

$$\|v\|^2_{\sigma,s,b} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (1 + |\tau + \xi^2|)^{2b}(1 + |\xi|)^{2s} e^{2\sigma(1+|\xi|)} |\hat{v}(\xi,\tau)|^2 \, d\xi d\tau.$$

The symbol in the $X_{\sigma,s,b}$-norm is adjusted to the linear part of the equation. In the following, some identities and linear estimates are listed that elucidate this relation. For the initial-value problem

$$\begin{aligned} &iw_t + w_{xx} = 0, \\ &w(x,0) = w_0(x), \end{aligned} \tag{2.1}$$

an explicit solution is given in terms of the propagator $S(t)$ by

$$w(x,t) = S(t)w_0 = c\int_{-\infty}^{\infty} e^{ix\xi} e^{-it\xi^2} \widehat{w}_0(\xi) \, d\xi. \tag{2.2}$$

For $\rho \in \mathbb{R}$, define the operator Λ^ρ by

$$\widehat{\Lambda^\rho v}(\xi,\tau) = (1 + |\tau|)^\rho \, \hat{v}(\xi,\tau).$$

Then we have the following identity:

$$\|S(t)v\|_{\sigma,s,b} = \|A^s e^{\sigma A}\Lambda^b v\|_{L^2(\mathbb{R}^2)}. \tag{2.3}$$

As is well known, the space $C([0,T], G^{\sigma,s})$ is a Banach space when equipped with the norm

$$|v|_{\mathcal{C}_{T,\sigma,s}} = \sup_{0 \le t \le T} \|v(\cdot,t)\|_{G^{\sigma,s}}.$$

For $b > \frac{1}{2}$, the space $X_{\sigma,s,b}$ is embedded in $C([0,T], G^{\sigma,s})$ as is evident from the inequality

$$|v|_{\mathcal{C}_{T,\sigma,s}} \le c\|v\|_{\sigma,s,b}, \tag{2.4}$$

which follows directly from (2.3) and the Sobolev Embedding Theorem.

In order to obtain local estimates in time, it is necessary to introduce a time cut-off function. Let ψ be an infinitely differentiable function on $\mathbb{R}$, such that

$$\psi(t) = \begin{cases} 0, & |t| \ge 2, \\ 1, & |t| \le 1, \end{cases}$$

and let $\psi_T(t) = \psi(t/T)$.

Lemma 1. *Let $\sigma > 0$, $b > \frac{1}{2}$, and $b-1 < b' < 0$. Then there is a constant c such that the following estimates hold.*

$$\|\psi(t)\, S(t)u_0(x)\|_{\sigma,s,b} \leq c\|u_0\|_{G^{\sigma,s}}, \tag{2.5}$$

$$\left\|\psi_T(t) \int_0^t S(t-s)v(s)\, ds\right\|_{\sigma,s,b} \leq cT^{1-b+b'}\|v\|_{\sigma,s,b'}. \tag{2.6}$$

Proof. With the same assumptions as in the lemma, but with $\sigma = 0$, (2.5) was proved in [19], and (2.6) was proved in [7]. These inequalities clearly remain valid for $\sigma > 0$, as one merely has to replace u_0 by $e^{\sigma A}u_0$ and v by $e^{\sigma A}v$. ■

Finally, note the following key result of Takaoka used in the proof of the nonlinear estimate (cf. [22], Lemma 3.1).

Lemma 2. *For $\frac{1}{2} \leq s < 1$, $b > \frac{1}{2}$ and $-b \leq b' \leq -\frac{3}{8}$, there holds*

$$\begin{aligned}
&\int_{\mathbb{R}^6} \frac{d(\xi,\tau)\ (1+|\xi|)^s}{\big(1+|\tau+\xi^2|\big)^{-b'}}\ \frac{f(\xi_1,\tau_1)\big(1+|\xi_1|\big)^{-s}}{\big(1+|\tau_1+\xi_1^2\big)^b} \\
&\quad \times \frac{g(\xi-\xi_2,\tau-\tau_2)\big(1+|\xi-\xi_2|\big)^{-s}}{\big(1+|\tau-\tau_2+(\xi-\xi_2)^2|\big)^b}\ \frac{h(\xi_2-\xi_1,\tau_2-\tau_1)\big(1+|\xi_2-\xi_1|\big)^{1-s}}{\big(1+|\tau_2-\tau_1-(\xi_2-\xi_1)^2|\big)^b}\, d\mu \\
&\quad \leq c\,\|d\|_{L^2_\xi L^2_\tau}\,\|f\|_{L^2_\xi L^2_\tau}\,\|g\|_{L^2_\xi L^2_\tau}\,\|h\|_{L^2_\xi L^2_\tau}.
\end{aligned}$$

3 Nonlinear estimates

In this section estimates for the nonlinear terms in the equation are proved. The first result will give control over the derivative term in equation (1.2).

Theorem 2. *Suppose u, v and w are in $X_{\sigma,s,b}$ where $s \geq \frac{1}{2}$, $b > \frac{1}{2}$, and $-b \leq b' \leq -\frac{3}{8}$. Then there exists a constant c depending only on s, b, and b' such that*

$$\|uv\bar{w}_x\|_{\sigma,s,b'} \leq c\,\|u\|_{\sigma,s,b}\,\|v\|_{\sigma,s,b}\,\|w\|_{\sigma,s,b}. \tag{3.1}$$

Proof. The proof of this theorem relies on Lemma 2. First note that (3.1) can be written more explicitly as

$$\left\|\big(1+|\tau+\xi^2|\big)^{b'}\big(1+|\xi|\big)^s e^{\sigma(1+|\xi|)}\widehat{uv\bar{w}_x}(\xi,\tau)\right\|_{L^2_\xi L^2_\tau} \leq c\,\|u\|_{\sigma,s,b}\,\|v\|_{\sigma,s,b}\,\|w\|_{\sigma,s,b}. \tag{3.2}$$

Now observe that if we let

$$\begin{aligned}
U^+(\xi,\tau) &= (1+|\tau+\xi^2|)^b\big(1+|\xi|\big)^s e^{\sigma(1+|\xi|)}\hat{u}(\xi,\tau), \\
V^+(\xi,\tau) &= (1+|\tau+\xi^2|)^b\big(1+|\xi|\big)^s e^{\sigma(1+|\xi|)}\hat{v}(\xi,\tau),
\end{aligned}$$

and

$$W^-(\xi,\tau) = (1+|\tau-\xi^2|)^b\big(1+|\xi|\big)^s e^{\sigma(1+|\xi|)}\hat{w}(-\xi,-\tau),$$

then (3.2) is equivalent to

$$\left\| \frac{\left(1+|\xi|\right)^{s} e^{\sigma(1+|\xi|)}}{(1+|\tau+\xi^2|)^{-b'}} \int_{\mathbb{R}^4} \frac{\left(1+|\xi_1|\right)^{-s} e^{-\sigma(1+|\xi_1|)} U^+(\xi_1,\tau_1)}{(1+|\tau_1+\xi_1^2|)^{b}} \right.$$
$$\times \frac{\left(1+|\xi-\xi_2|\right)^{-s} e^{-\sigma(1+|\xi-\xi_2|)} V^+(\xi-\xi_2,\tau-\tau_2)}{(1+|\tau-\tau_2+(\xi-\xi_2)^2|)^{b}}$$
$$\left. \times \frac{\left(1+|\xi_2-\xi_1|\right)^{1-s} e^{-\sigma(1+|\xi_2-\xi_1|)} W^-(\xi_2-\xi_1,\tau_2-\tau_1)}{\left(1+|\tau_2-\tau_1-(\xi_2-\xi_1)^2|\right)^{b}} \, d\xi_2 d\tau_2 d\xi_1 d\tau_1 \right\|_{L^2_\xi L^2_\tau}$$
$$\leq c \, \|U^+\|_{L^2_\xi L^2_\tau} \, \|V^+\|_{L^2_\xi L^2_\tau} \, \|W^-\|_{L^2_\xi L^2_\tau}.$$

A proof of this estimate can be obtained by duality. Let $d(\xi,\tau)$ be a positive function in $L^2(\mathbb{R}^2)$ with norm $\|d\|_{L^2(\mathbb{R}^2)} = 1$. Then we need to estimate

$$\int_{\mathbb{R}^6} \frac{d(\xi,\tau)\ (1+|\xi|)^s\ e^{\sigma(1+|\xi|)}}{\left(1+|\tau+\xi^2|\right)^{-b'}} \ \frac{U^+(\xi_1,\tau_1)\ e^{-\sigma(1+|\xi_1|)} \left(1+|\xi_1|\right)^{-s}}{\left(1+|\tau_1+\xi_1^2\right)^{b}}$$
$$\times \frac{V^+(\xi-\xi_2,\tau-\tau_2)\, e^{-\sigma(1+|\xi-\xi_2|)} \left(1+|\xi-\xi_2|\right)^{-s}}{\left(1+|\tau-\tau_2+(\xi-\xi_2)^2|\right)^{b}}$$
$$\times \frac{W^-(\xi_2-\xi_1,\tau_2-\tau_1)\, e^{-\sigma(1+|\xi_2-\xi_1|)} \left(1+|\xi_2-\xi_1|\right)^{1-s}}{\left(1+|\tau_2-\tau_1-(\xi_2-\xi_1)^2|\right)^{b}} \, d\mu,$$

where $d\mu = d\xi_2 d\tau_2 d\xi_1 d\tau_1 d\xi d\tau$. Using the triangle inequality $|\xi| \leq |\xi_1| + |\xi-\xi_2| + |\xi_2-\xi_1|$ on the exponential terms leads to

$$\int_{\mathbb{R}^6} \frac{d(\xi,\tau) \left(1+|\xi|\right)^s}{\left(1+|\tau+\xi^2|\right)^{-b'}} \ \frac{U^+(\xi_1,\tau_1) \left(1+|\xi_1|\right)^{-s}}{\left(1+|\tau_1+\xi_1^2\right)^{b}} \ \frac{V^+(\xi-\xi_2,\tau-\tau_2) \left(1+|\xi-\xi_2|\right)^{-s}}{\left(1+|\tau-\tau_2+(\xi-\xi_2)^2|\right)^{b}}$$
$$\times \frac{W^-(\xi_2-\xi_1,\tau_2-\tau_1) \left(1+|\xi_2-\xi_1|\right)^{1-s}}{\left(1+|\tau_2-\tau_1-(\xi_2-\xi_1)^2|\right)^{b}} \, d\mu.$$

According to Lemma 2, this integral can be dominated by

$$c \, \|d\|_{L^2_\xi L^2_\tau} \, \|U^+\|_{L^2_\xi L^2_\tau} \, \|V^+\|_{L^2_\xi L^2_\tau} \, \|W^-\|_{L^2_\xi L^2_\tau}. \qquad \blacksquare$$

The next result provides control on the quintic term in the equation.

Theorem 3. *Suppose u, v, and w are in $X_{\sigma,s,b}$ where $s > \frac{1}{2}$, $b > \frac{1}{2}$, and $b' < -\frac{1}{4}$. Then there exists a constant c depending only on s, b, and b' such that*

$$\|u\bar{u}v\bar{v}w\|_{\sigma,s,b'} \leq c \, \|u\|^2_{\sigma,s,b} \, \|v\|^2_{\sigma,s,b} \, \|w\|_{\sigma,s,b}. \tag{3.3}$$

In order to prove this theorem, several auxiliary results are needed. First, we introduce the following notation. For $\rho \in \mathbb{R}$, and a suitable function f, define F_ρ by its Fourier transform $\widehat{F}_\rho$,

$$\widehat{F}_\rho(\xi,\tau) = \frac{f(\xi,\tau)}{\left(1+|\tau+\xi^2|\right)^{\rho}}. \tag{3.4}$$

The first lemma is similar to a lemma proved by Bourgain [2], and we refer the reader to his article for the proof.

Lemma 3. *For $\rho > \frac{1}{4}$, there exists a constant c depending only on ρ such that*

$$\|F_\rho\|_{L^4_x L^2_t} \le c\|f\|_{L^2_\xi L^2_\tau}. \tag{3.5}$$

Lemma 4. *For $\rho > \frac{1}{2}$, and $s > \frac{1}{2}$, there exists a constant c depending only on ρ and s, such that*

$$\|A^{-s} F_\rho\|_{L^\infty_x L^\infty_t} \le c\|f\|_{L^2_\xi L^2_\tau}. \tag{3.6}$$

This lemma follows directly from the Sobolev embedding theorem (cf. [8]). The following proposition which is the basis for the proof of the next lemma is due to Kenig and Ruiz [21]. It can also be found in [20].

Proposition 1. *Let $S(t)$ be the propagator for the linear Schrödinger equation as introduced in (2.2). Then for any $u_0 \in H^{\frac{1}{4}}$,*

$$\left\{\int_{-\infty}^{\infty} \sup_{-\infty<t<\infty} |S(t)u_0(x)|^4 dx\right\}^{\frac{1}{4}} \le c\,\|A^{\frac{1}{4}} u_0\|_{L^2_x}. \tag{3.7}$$

Lemma 5. *For $\rho > \frac{1}{2}$ and $s \ge \frac{1}{4}$,*

$$\|A^{-s} F_\rho\|_{L^4_x L^\infty_t} \le c\|f\|_{L^2_\xi L^2_\tau}. \tag{3.8}$$

Proof. The proof follows an argument given in [17]. Writing $A^{-s}F_\rho$ in terms of its Fourier transform and then changing variables to $\lambda = \tau + \xi^2$, we obtain

$$A^{-s} F_\rho(x,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ix\xi} e^{it\tau} \frac{f(\xi,\tau)}{(1+|\xi|)^s\big(1+\left|\tau+\xi^2\right|\big)^\rho}\, d\tau\, d\xi \tag{3.9}$$

$$= \int_{-\infty}^{\infty} e^{it\lambda}\left(\int_{-\infty}^{\infty} e^{ix\xi} e^{it\xi^2} \frac{f(\xi,\lambda+\xi^2|)}{(1+|\xi|)^s}\, d\xi\right) \frac{d\lambda}{(1+|\lambda|)^\rho}. \tag{3.10}$$

Now defining $u_\lambda(x)$ as the inverse Fourier transform of $f(\xi,\lambda+\xi^2)$ in the x-variable, using Minkowski's integral inequality, Proposition 1, and finally Hölder's inequality, we can write

$$\begin{aligned}
\|A^{-s}F_\rho\|_{L^4_x L^\infty_t} &\le \int_{-\infty}^{\infty} \left\|\int_{-\infty}^{\infty} e^{ix\xi} e^{it\xi^2} \frac{f(\xi,\lambda+\xi^2)}{(1+|\xi|)^s}\, d\xi\right\|_{L^4_x L^\infty_t} \frac{d\lambda}{(1+|\lambda|)^\rho} \\
&\le c \int_{-\infty}^{\infty} \|\, S(t)A^{-s}u_\lambda\|_{L^4_x L^\infty_t} \frac{d\lambda}{(1+|\lambda|)^\rho} \\
&\le c \int_{-\infty}^{\infty} \|u_\lambda\|_{L^2_x} \frac{d\lambda}{(1+|\lambda|)^\rho} \\
&= c\,\|f\|_{L^2_\xi L^2_\tau}.
\end{aligned}$$

■

With these estimates in hand, Theorem 3 can be proved.

Proof. First, let

$$U^-(\xi,\tau) = (1+|\tau-\xi^2|)^b\big(1+|\xi|\big)^s e^{\sigma(1+|\xi|)}\hat{u}(-\xi,-\tau),$$
$$V^-(\xi,\tau) = (1+|\tau-\xi^2|)^b\big(1+|\xi|\big)^s e^{\sigma(1+|\xi|)}\hat{v}(-\xi,-\tau),$$

and

$$W^+(\xi,\tau) = (1+|\tau+\xi^2|)^b\big(1+|\xi|\big)^s e^{\sigma(1+|\xi|)}\hat{w}(\xi,\tau).$$

Then similarly to the proof of Theorem 2, we have to estimate an integral of the form

$$\begin{aligned}\int_{\mathbb{R}^{10}} &\frac{d(\xi,\tau)\,\big(1+|\xi|\big)^s\,e^{\sigma(1+|\xi|)}}{\big(1+|\tau+\xi^2|\big)^{-b'}}\,\frac{U^+(\xi_1,\tau_1)\,e^{-\sigma(1+|\xi_1|)}\big(1+|\xi_1|\big)^{-s}}{\big(1+|\tau_1+\xi_1^2|\big)^b}\\ &\times\frac{U^-(\xi-\xi_4,\tau-\tau_4)\,e^{-\sigma(1+|\xi-\xi_4|)}\big(1+|\xi-\xi_4|\big)^{-s}}{\big(1+|\tau-\tau_4-(\xi-\xi_4)^2|\big)^b}\\ &\times\frac{V^+(\xi_4-\xi_3,\tau_4-\tau_3)\,e^{-\sigma(1+|\xi_4-\xi_3|)}\big(1+|\xi_4-\xi_3|\big)^{-s}}{\big(1+|\tau_4-\tau_3+(\xi_4-\xi_3)^2|\big)^b}\\ &\times\frac{V^-(\xi_3-\xi_2,\tau_3-\tau_2)\,e^{-\sigma(1+|\xi_3-\xi_2|)}\big(1+|\xi_3-\xi_2|\big)^{-s}}{\big(1+|\tau_3-\tau_2-(\xi_3-\xi_2)^2|\big)^b}\\ &\times\frac{W^+(\xi_2-\xi_1,\tau_2-\tau_1)\,e^{-\sigma(1+|\xi_2-\xi_1|)}\big(1+|\xi_2-\xi_1|\big)^{-s}}{\big(1+|\tau_2-\tau_1+(\xi_2-\xi_1)^2|\big)^b}\,d\mu,\end{aligned}$$

where now $d\mu = d\xi_4 d\tau_4 d\xi_3 d\tau_3 d\xi_2 d\tau_2 d\xi_1 d\tau_1 d\xi d\tau$. Using the inequality $|\xi| \le |\xi_1| + |\xi-\xi_4| + |\xi_4-\xi_3| + |\xi_3-\xi_2| + |\xi_2-\xi_1|$ on the exponentials, we are left with

$$\begin{aligned}\int_{\mathbb{R}^{10}} &\frac{d(\xi,\tau)\,|\xi|^s}{\big(1+|\tau+\xi^2|\big)^{-b'}}\,\frac{U^+(\xi_1,\tau_1)\big(1+|\xi_1|\big)^{-s}}{\big(1+|\tau_1+\xi_1^2|\big)^b}\,\frac{U^-(\xi-\xi_4,\tau-\tau_4)\,\big(1+|\xi-\xi_4|\big)^{-s}}{\big(1+|\tau-\tau_4-(\xi-\xi_4)^2|\big)^b}\\ &\times\frac{V^+(\xi_4-\xi_3,\tau_4-\tau_3)\,\big(1+|\xi_4-\xi_3|\big)^{-s}}{\big(1+|\tau_4-\tau_3+(\xi_4-\xi_3)^2|\big)^b}\,\frac{V^-(\xi_3-\xi_2,\tau_3-\tau_2)\,\big(1+|\xi_3-\xi_2|\big)^{-s}}{\big(1+|\tau_3-\tau_2-(\xi_3-\xi_2)^2|\big)^b}\\ &\times\frac{W^+(\xi_2-\xi_1,\tau_2-\tau_1)\,\big(1+|\xi_2-\xi_1|\big)^{-s}}{\big(1+|\tau_2-\tau_1+(\xi_2-\xi_1)^2|\big)^b}\,d\mu.\end{aligned}$$

Now, split the Fourier space into 24 regions, according to all possible combinations of inequalities such as $|\xi-\xi_4| \le |\xi_4-\xi_3| \le |\xi_3-\xi_2| \le |\xi_2-\xi_1| \le |\xi_1|$. In this particular case, the integral can be dominated by

$$\langle D_{-b'}\,,\,U_b^+\;A^{-s}U_b^-\;A^{-s}V_b^+\;A^{-s}V_b^-\;A^{-s}W_b^+\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2(\mathbb{R}^2)$, and $D_{-b'}$, U_b^+, etc. are defined as in (3.4).

Then the estimate continues as follows:

$$
\begin{aligned}
&\langle D_{-b'}\,,\, U_b^+ \; A^{-s}U_b^- \; A^{-s}V_b^+ \; A^{-s}V_b^- \; A^{-s}W_b^+\rangle \\
&\qquad \le c\,\|D_{-b'}\|_{L_x^4L_t^2}\,\|U_b^+\|_{L_x^4L_t^2}\,\|A^{-s}U_b^-\|_{L_x^4L_t^\infty} \\
&\qquad\quad \times \|A^{-s}V_b^+\|_{L_x^4L_t^\infty}\,\|A^{-s}V_b^-\|_{L_x^\infty L_t^\infty}\,\|A^{-s}W_b^+\|_{L_x^\infty L_t^\infty} \\
&\qquad \le c\,\|d\|_{L_\xi^2L_\tau^2}\,\|u\|^2_{L_\xi^2L_\tau^2}\,\|v\|^2_{L_\xi^2L_\tau^2}\,\|w\|_{L_\xi^2L_\tau^2} \\
&\qquad = c\,\|u\|^2_{L_\xi^2L_\tau^2}\,\|v\|^2_{L_\xi^2L_\tau^2}\,\|w\|_{L_\xi^2L_\tau^2}.
\end{aligned}
$$

The other cases follow simply by interchanging the roles of U_b^+, U_b^-, V_b^+, V_b^-, and W_b^+. ■

The estimates in Theorems 2 and 3, coupled with the linear estimates in the previous section suffice to prove the local well-posedness of (1.2) in the analytic classes $X_{\sigma,s,b}$, as is shown in the next section.

4 Proof of Theorem 1

With the estimates provided in the previous section, existence and uniqueness of a solution to the initial-value problem in $X_{\sigma,s,b}$ can be proved easily using a contraction argument. First we prove existence of the function v satisfying equation (1.2). Consider the integral operator

$$
\Gamma(v) = \psi(t)S(t)v_0 - \psi_T(t)\int_0^t S(t-t')\left(iv^2\bar{v}_x + \tfrac{1}{2}|v|^4v\right)\,dt'. \tag{4.1}
$$

Let $r = \|v_0\|_{G^{\sigma,s}}$. It will be proved that T can be chosen so that Γ is a contraction in the ball $B(2cr) \subset X_{\sigma,s,b}$ of radius $2cr$ centered at 0.

Lemma 6. *There exists a positive time T, such that the operator Γ as defined in (4.1) is a contraction in the ball $B(2cr)$.*

Proof. First it is proved that Γ is a mapping on $B(2cr)$. Using (2.5) and (2.6), and the nonlinear estimates, we see that

$$
\begin{aligned}
\|\Gamma(v)\|_{\sigma,s,b} &\le \|\psi(t)S(t)v_0\|_{\sigma,s,b} + \left\|\psi_T(t)\int_0^t S(t-t')v(\cdot,t')^2\bar{v}_x(\cdot,t')\,dt'\right\|_{\sigma,s,b} \\
&\quad + \left\|\psi_T(t)\int_0^t S(t-t')|v(\cdot,t')|^4v(\cdot,t')\,dt'\right\|_{\sigma,s,b} \\
&\le c\,\|v_0\|_{G^{\sigma,s}} + c,\; T^{1-b+b'}\,\|v^2\partial_x\bar{v}\|_{\sigma,s,b'} + c\,T^{1-b+b'}\,\||v|^4v\|_{\sigma,s,b'} \\
&\le c\,\|v_0\|_{G^{\sigma,s}} + c^2\,T^{1-b+b'}\,\|v\|^3_{\sigma,s,b} + c^2\,T^{1-b+b'}\,\|v\|^5_{\sigma,s,b} \\
&\le cr + c^2\,T^{1-b+b'}\left((2cr)^3 + (2cr)^5\right) \\
&\le cr + 2c^2\,T^{1-b+b'}\left(2c(r+1)\right)^5.
\end{aligned}
$$

It can now be seen that T may be chosen in such a way that Γ maps $B(2cr)$ to $B(2cr)$. A similar argument (cf. [8]) proves the contraction property. ■

Since Γ is a contraction, it follows that Γ has a unique fixed point v in $B(2cr)$. The function v solves the initial-value problem (1.2). Following standard arguments as in [8], the persistence property, as well as uniqueness and and continuous dependence on initial data in $C([0,T], G^{\sigma,s})$ can be obtained. The local well-posedness of the original problem (1.1) follows immediately from the continuity of the gauge transformation and its inverse in $G^{\sigma,s}$.

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