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Space Analyticity for the Navier–Stokes and Related Equations with Initial Data in L^p

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We introduce a method of estimating the space analyticity radius of solutions for the Navier–Stokes and related equations in terms of L^p and L^∞ norms of the initial data. The method enables us to express the space analyticity radius for 3D Navier–Stokes equations in terms of the Reynolds number of the flow. Also, for the Kuramoto–Sivashinsky equation, we give a partial answer to a conjecture that the radius of space analyticity on the attractor is independent of the spatial period.

1. INTRODUCTION

In [FT], Foias and Temam introduced a method for estimating the space analyticity radius of solutions of the Navier–Stokes equations (NSE). The basic idea of interpolating between a suitably defined analyticity norm and a Sobolev norm leads to a very simple energy method which eliminates the need of traditional estimates on the higher order derivatives (as in e.g. [M]). The method is applicable to many other equations (c.f. [CEES2], [G], etc.); however, due to a use of Fourier series expansions, it is a Hilbert space method and it is not suitable for L^p initial data.

In this paper, we introduce a method which bridges this difficulty and offers a simple estimate of the analyticity radius in terms of the L^p norm of the initial data. Instead of estimating a priori the analyticity norm $\|e^{\alpha t A^{1/2}} u\|$, we base our considerations on interpolating between the L^p

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norm of the initial datum and the L^p norm of the complexified solution. The presented method is suitable for a rather singular initial data, and it extends easily to other semilinear equations of elliptic or parabolic type. Dirichlet boundary conditions will be considered in our forthcoming paper.

The paper is organized in the following way: In Section 2, with the main result contained in Theorem 2.1, we present the method for the NSE with initial data in $L^p(\mathbb{R}^D)$. In Section 3 (c.f. Theorem 3.1), we address the periodic boundary conditions. In Theorem 3.6, we consider bounded initial data. Thus we are able to express the real-analyticity radius of a solution in terms of the Reynolds number. In Section 4 (c.f. Theorem 4.1), we consider the Kuramoto–Sivashinsky equation with bounded initial data. In particular, Theorem 4.4 provides a partial answer to a conjecture from [CEES2].

For some other methods for establishing real-analyticity for solutions of evolution equations, c.f. [KM], [TBDVT].

2. THE NAVIER–STOKES EQUATIONS IN $\mathbb{R}^D$

The NSE in $\mathbb{R}^D (D \geqslant 2)$ read as

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u + (u \cdot \nabla)\, u + \nabla \pi = f \\ \nabla \cdot u = 0, \end{aligned} \tag{2.1}$$

where the force $f\colon \mathbb{R}^D \times [0, \infty) \to \mathbb{R}^D$ is given, while the pressure $\pi\colon \mathbb{R}^D \times [0, \infty) \to \mathbb{R}$ and the velocity $u\colon \mathbb{R}^D \times [0, \infty) \to \mathbb{R}^D$ are unknown. We also impose the initial condition

$$u(x, 0) = u_0(x), \qquad x \in \mathbb{R}^D, \tag{2.2}$$

where the initial velocity $u_0 \in L^p(\mathbb{R}^D)^D$ is given and assumed to be divergence-free. On the force f we impose the following assumption: Assume that $f(\cdot, t)$ is divergence-free and real-analytic in the space variable with the analyticity radius at least $\delta_f > 0$ for all $t \in [0, \infty)$. Let $f + ig$ be its analytic extension, and assume, for simplicity, that

$$f, g \in C^\infty(\{z \in \mathbb{C}^n : |\mathrm{Im}\, z| < \delta_f\} \times [0, \infty)) \tag{2.3}$$

and

$$\gamma = \sup_{t \geqslant 0} \sup_{|y| < \delta_f} (\| f(\cdot, y, t)\|_{L^p} + \|g(\cdot, y, t)\|_{L^p}) < \infty. \tag{2.4}$$

A function u is a solution of (2.1), (2.2) for $t \in [0, T)$ if the following holds: we have $u \in C([0, T), L^p(\mathbb{R}^D)^D)$ with $u(0) = u_0$, and there exists $\pi \in C([0, T), L^p(\mathbb{R}^D))$ such that u and π are classical solutions of (2.1) for $t \in (0, T)$.

For properties of solutions of the NSE in $\mathbb{R}^D$, the reader is referred to [FJR] and [K]. Here we recall only the following existence result. Let $u_0 \in L^p(\mathbb{R}^D)^D$, where $p \in (D, \infty)$, be divergence free. Then there exists $T > 0$, depending only on p, D, and $\|u_0\|_{L^p}$, and a solution u (with the associated π) of the NSE for $t \in [0, T)$. Moreover,

$$\Delta \pi = -\partial_j \partial_k (u_j u_k)$$

for $t \geqslant 0$, where $u = (u_1, ..., u_D)$ and where the summation convention is used.

For simplicity, we assume $p \in (D, \infty)$, although the case $p = D$ could also be treated using ideas from [K]. The symbol C below will denote a positive constant depending only on dimension D; it may also depend on p, but only as $p \to D$. The following is the main result of this section.

THEOREM 2.1. *Assume* $\|u_0\|_{L^p} \leqslant M_p < \infty$, *and let*

$$T = \min \left\{ \frac{1}{C p^2 M_p^{2p/(p-D)}}, \frac{M_p}{C\gamma} \right\}, \tag{2.5}$$

where γ *is defined in* (2.4). *Then there exists a solution* $u \in C([0, T), L^p(\mathbb{R}^D)^D)$ *of the NSE with the following property*: *For every* $t \in (0, T)$, u *is a restriction of an analytic function* $u(x, y, t) + iv(x, y, t)$ *in the region*

$$\mathcal{D}_t = \{(x, y) \in \mathbb{C}^D : |y| \leqslant C^{-1} t^{1/2}, |y| < \delta_f\};$$

moreover,

$$\|u(\cdot, y, t)\|_{L^p} + \|v(\cdot, y, t)\|_{L^p} \leqslant C M_P \tag{2.6}$$

for $t \in (0, T)$ *and* $(x, y) \in \mathcal{D}_t$.

In order to solve the NSE, we form a sequence of approximating solutions $u^{(n)}$ and $\pi^{(n)}$ obtained in the following way. We set $u^{(0)} \equiv 0$, $\pi^{(0)} \equiv 0$, and then construct sequences of functions $u^{(n)} \in C([0, \infty), L^p(\mathbb{R}^D)^D)$ and $\pi^{(n)} \in C([0, \infty), L^p(\mathbb{R}^D))(n = 1, 2, ...)$ such that

$$\partial_t u^{(n)} - \Delta u^{(n)} = -(u^{(n-1)} \cdot \nabla) u^{(n-1)} - \nabla \pi^{(n-1)} + f$$

and

$$\Delta \pi^{(n)} = -\partial_j \partial_k (u_j^{(n)} u_k^{(n)})$$

with the initial condition

$$u^{(n)}(x, 0) = u_0(x), \qquad x \in \mathbb{R}^D.$$

It is well-known (c.f. [K]) that $u^{(n)}$ and $\pi^{(n)}$ converge to a solution of the NSE with the initial datum u_0 on some interval $[0, \varepsilon)$ with $\varepsilon > 0$; however, this fact also follows from our proof below.

By the well-known analyticity properties of the heat and the Laplace equations, $u^{(n)}$ and $\pi^{(n)}$ are real-analytic with the real-analyticity radius δ_f for every $t \in (0, \infty)$. Let $u^{(n)} + iv^{(n)}$ and $\pi^{(n)} + i\rho^{(n)}$ be the analytic extensions of $u^{(n)}$ and $\pi^{(n)}$ respectively. Then

$$\begin{aligned} \partial_t u^{(n)} - \Delta u^{(n)} &= -(u^{(n-1)} \cdot \nabla)\, u^{(n-1)} + (v^{(n-1)} \cdot \nabla)\, v^{(n-1)} - \nabla \pi^{(n-1)} + f \\ \partial_t v^{(n)} - \Delta v^{(n)} &= -(u^{(n-1)} \cdot \nabla)\, v^{(n-1)} + (v^{(n-1)} \cdot \nabla)\, u^{(n-1)} - \nabla \rho^{(n-1)} + g \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} \Delta \pi^{(n)} &= -\partial_{jk}(u_j^{(n)} u_k^{(n)} - v_j^{(n)} v_k^{(n)}) \\ \Delta \rho^{(n)} &= -2\, \partial_{jk}(u_j^{(n)} v_k^{(n)}) \end{aligned} \tag{2.8}$$

for $t \in (0, \infty)$. We proceed to estimate

$$\phi^{(n)}(t) = \|u^{(n)}(\cdot, \alpha t, t)\|_{L^p} + \|v^{(n)}(\cdot, \alpha t, t)\|_{L^p}$$

where $t \geq 0$ and $\alpha \in \mathbb{R}^D$. Note that

$$\phi^{(n)}(0) = \|u_0\|_{L^p}.$$

It is convenient to denote

$$\begin{aligned} U_\alpha^{(n)}(x, t) &= u^{(n)}(x, \alpha t, t) \\ V_\alpha^{(n)}(x, t) &= v^{(n)}(x, \alpha t, t) \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} \Pi_\alpha^{(n)}(x, t) &= \pi^{(n)}(x, \alpha t, t) \\ R_\alpha^{(n)}(x, t) &= \rho^{(n)}(x, \alpha t, t) \end{aligned} \tag{2.10}$$

for $n \in \mathbb{N}_0 = \{0, 1, ...\}$, $x, \alpha \in \mathbb{R}^D$, and $t \geq 0$; also, let

$$\begin{aligned} F_\alpha(x, t) &= f(x, \alpha t, t) \\ G_\alpha(x, t) &= g(x, \alpha t, t). \end{aligned} \tag{2.11}$$

Then

$$\partial_t U^{(n)} - \Delta U^{(n)} = -\alpha_j \partial_j V^{(n)} - (U^{(n-1)} \cdot \nabla) U^{(n-1)} + (V^{(n-1)} \cdot \nabla) V^{(n-1)} - \nabla \Pi^{(n-1)} + F$$

and

$$\partial_t V^{(n)} - \Delta V^{(n)} = \alpha_j \partial_j U^{(n)} - (U^{(n-1)} \cdot \nabla) V^{(n-1)} + (V^{(n-1)} \cdot \nabla) U^{(n-1)} - \nabla R^{(n-1)} + G$$

where the subscript α is omitted for simplicity, with initial conditions

$$U^{(n)}(x, 0) = u_0(x)$$

$$V^{(n)}(x, 0) = 0$$

for $x \in \mathbb{R}^D$; also,

$$\begin{aligned} \Delta \Pi^{(n)} &= -\partial_{jk}(U_j^{(n)} U_k^{(n)} - V_j^{(n)} V_k^{(n)}) \\ \Delta R^{(n)} &= -2\, \partial_{jk}(U_j^{(n)} V_k^{(n)}). \end{aligned} \tag{2.12}$$

Hence,

$$\begin{aligned} U^{(n)}(x, t) = &\int \Gamma(x-w, t)\, u_0(w)\, dw \\ &- \int_0^t \int \partial_j \Gamma(x-w, t-s)\, U_j^{(n-1)}(w, s)\, U^{(n-1)}(w, s)\, dw\, ds \\ &+ \int_0^t \int \partial_j \Gamma(x-w, t-s)\, V_j^{(n-1)}(w, s)\, V^{(n-1)}(w, s)\, dw\, ds \\ &- \int_0^t \int \nabla \Gamma(x-w, t-s)\, \Pi^{(n-1)}(w, s)\, dw\, ds \\ &+ \int_0^t \int \Gamma(x-w, t-s)\, F(w, s)\, dw\, ds \\ &- \alpha_j \int_0^t \int \partial_j \Gamma(x-w, t-s)\, V^{(n)}(w, s)\, dw\, ds \end{aligned} \tag{2.13}$$

(the integrals in w being taken over $\mathbb{R}^D$) and

$$\begin{aligned} V^{(n)}(x,t) = &-\int_0^t \int \partial_j \Gamma(x-w,t-s)\, U_j^{(n-1)}(w,s)\, V^{(n-1)}(w,s)\, dw\, ds \\ &-\int_0^t \int \partial_j \Gamma(x-w,t-s)\, V_j^{(n-1)}(w,s)\, U^{(n-1)}(w,s)\, dw\, ds \\ &-\int_0^t \int \nabla \Gamma(x-w,t-s)\, R^{(n-1)}(w,s)\, dw\, ds \\ &+\int_0^t \int \Gamma(x-w,t-s)\, G(w,s)\, dw\, ds \\ &+\alpha_j \int_0^t \int \partial_j \Gamma(x-w,t-s)\, U^{(n)}(w,s)\, dw\, ds, \end{aligned} \tag{2.14}$$

where

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{D/2}} \exp\left(-\frac{|x|^2}{4t}\right)$$

is the Gaussian kernel. We will use the following well-known estimates.

LEMMA 2.2. *We have* $\|\Gamma(\cdot,t)\|_{L^1} \leqslant 1$ *for* $t > 0$ *and*

$$\|\nabla \Gamma\|_{L^{q,1}(S_T)} \leqslant C(q)\, T^{(q+D-Dq)/2q}, \qquad 1 \leqslant q < \frac{D}{D-1},$$

where $S_T = \mathbb{R}^D \times [0,T]$ *with* $T > 0$.

Proof. This follows the identity $\Gamma(x,t) = (1/t^{D/2})\, \Gamma(x/t^{1/2},1)$ for $x \in \mathbb{R}^D$ and $t > 0$. ∎

Above we used the notation

$$\|u\|_{L^{p,q}(S_T)} = \left(\int_0^T \|u(\cdot,t)\|_{L^p}^q\, dt\right)^{1/q}$$

with the usual agreement when $q = \infty$.

Using Lemma 2.2 and the Calderón–Zygmund theorem ([S]) for the equation (2.12), we get for $p \in (D,\infty)$

$$\begin{aligned} \|U^{(n)}\|_{L^{p,\infty}(S_T)} \leqslant\ & \|u_0\|_{L^p} + CpT^{1/2-D/2+D(p-1)/2p} \\ & \times (\|U^{(n-1)}\|^2_{L^{p,\infty}(S_T)} + \|V^{(n-1)}\|^2_{L^{p,\infty}(S_T)}) \\ & + CT\,\|F\|_{L^{p,\infty}(S_T)} + C\,|\alpha|\, T^{1/2}\, \|V^{(n)}\|_{L^{p,\infty}(S_T)} \end{aligned}$$

and

$$\|V^{(n)}\|_{L^{p,\infty}(S_T)} \leqslant CpT^{1/2-D/2+D(p-1)/2p} \|U^{(n-1)}\|_{L^{p,\infty}(S_T)} \|V^{(n-1)}\|_{L^{p,\infty}(S_T)} + CT\|G\|_{L^{p,\infty}(S_T)} + C|\alpha|\, T^{1/2} \|U^{(n)}\|_{L^{p,\infty}(S_T)}.$$

Assume

$$C|\alpha| T^{1/2} \leqslant \tfrac{1}{2}$$

for C as in the previous equations. Then

$$\|U^{(n)}\|_{L^{p,\infty}(S_T)} + \|V^{(n)}\|_{L^{p,\infty}(S_T)} \leqslant C\|u_0\|_{L^p} + CpT^{1/2-D/2+D(p-1)/2p}(\|U^{(n-1)}\|_{L^{p,\infty}(S_T)} + \|V^{(n-1)}\|_{L^{p,\infty}(S_T)})^2 + CT(\|F\|_{L^{p,\infty}(S_T)} + \|G\|_{L^{p,\infty}(S_T)}).$$

From here, we obtain by induction

$$\|U^{(n)}\|_{L^{p,\infty}(S_T)} + \|V^{(n)}\|_{L^{p,\infty}(S_T)} \leqslant CM_p \tag{2.15}$$

provided

$$0 < T \leqslant \min\left\{\frac{1}{C(pM_p)^{2p/(p-D)}},\ \frac{M_p}{C(\|F\|_{L^{p,\infty}(S_T)} + \|G\|_{L^{p,\infty}(S_T)})}\right\}.$$

Therefore, we conclude the following.

Lemma 2.3. *Assume*

$$|y| < \min\{\delta_f, C^{-1}T^{1/2}\} \tag{2.16}$$

and (2.5). *Then*

$$\|u^{(n)}(\cdot, y, t)\|_{L^p} + \|v^{(n)}(\cdot, y, t)\|_{L^p} \leqslant CM_p \tag{2.17}$$

for all $n \in \mathbb{N}$ *and* $t \in [0, T)$.

Proof. This immediately follows from the above estimates by setting $y = \alpha t$. ∎

In order to prove Theorem 2.1, we need the following simple lemma.

LEMMA 2.4. *Let $\mathscr{F}$ be the set of all functions f which are analytic in an open set $\Omega \subseteq \mathbb{C}^D$ and for which*

$$\int_\Omega |f(x, y)|^p \, dx \, dy \leqslant M_0 < \infty.$$

Then $\mathscr{F}$ is a normal family.

Proof. Note that every $f \in \mathscr{F}$ is harmonic in Ω. ∎

Now, we are ready to prove the main result.

Proof of Theorem 2.1. First, $u^{(n)}$ and $\pi^{(n)}$ converge to $u \in C([0, T), L^p(\mathbb{R}^D)^D)$ and $\pi \in C([0, T), L^p(\mathbb{R}^D))$, where, T is as in (2.5), such that $u(0) = u_0$. Indeed, for $n \in \{2, 3, ...\}$, we have

$$\begin{aligned} \|u^{(n)} - u^{(n-1)}\|_{L^{p,\infty}(S_T)} &\leqslant CpT^{1/2 - D/2 + D(p-1)/2p} (\|u^{(n-1)}\|_{L^{p,\infty}(S_T)} + \|u^{(n-2)}\|_{L^{p,\infty}(S_T)}) \\ &\quad \times \|u^{(n-1)} - u^{(n-2)}\|_{L^{p,\infty}(S_T)} \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} \|u^{(n)} - u_{\text{lin}}\|_{L^{p,\infty}(S_T)} \leqslant CpT^{1/2 - D/2 + D(p-1)/2p} \, \|u^{(n-1)}\|^2_{L^{p,\infty}(S_T)} \\ + CT \, \|f\|_{L^{p,\infty}(S_T)}, \end{aligned} \tag{2.19}$$

where

$$u_{\text{lin}}(x, t) = \int \Gamma(x - w, t) \, u_0(w) \, dw.$$

The fact then follows from the estimate (2.17) and the contraction mapping principle.

It remains to be shown that u and π are classical solutions of the NSE for $t \in (0, T)$ and that u has the required analyticity properties.

Denote by $\mathscr{D}$ the set of (x, y, t) such that $t \in (0, T)$, where T satisfies (2.5), and such that (2.16) holds. For every compact set $\mathscr{K} \subseteq \mathscr{D}$ and every $\alpha \in \mathbb{N}_0^D$ and $k \in \mathbb{N}_0$, the derivatives $\partial_t^k \, \partial_x^\alpha (u^{(n)} + iv^{(n)})$ and $\partial_x^\alpha (\pi^{(n)} + ip^{(n)})$ are uniformly bounded. Indeed, we may bound the space derivatives using Lemmas 2.3 and 2.4, while the time derivatives can be bounded using the equations (2.7) and (2.8). Therefore, there exist u, v, $\tilde{u}$, $\tilde{v} \in C^\infty(\mathscr{D})$ and π,

ρ, $\tilde{\pi}$, $\tilde{\rho} \in C^\infty(\mathscr{D})$ and an increasing sequence $\{n_k\}_{k=1}^\infty$ such that for every $\alpha \in \mathbb{N}_0^D$ and $k \in \mathbb{N}_0$

$$\partial_t^k \partial_x^\alpha (u^{(n_k)} + iv^{(n_k)}) \to \partial_t^k \partial_x^\alpha (u + iv)$$

$$\partial_x^\alpha (\pi^{(n_k)} + i\rho^{(n_k)}) \to \partial_x^\alpha (\pi + i\rho)$$

and

$$\partial_t^k \partial_x^\alpha (u^{(n_k+1)} + iv^{(n_k+1)}) \to \partial_t^k \partial_x^\alpha (\tilde{u} + i\tilde{v})$$

$$\partial_x^\alpha (\pi^{(n_k+1)} + i\rho^{(n_k+1)}) \to \partial_x^\alpha (\tilde{\pi} + i\tilde{\rho})$$

uniformly on compact subsets of $\mathscr{D}$. Clearly, (u, π) and $(\tilde{u}, \tilde{\pi})$ agree with (u, π) from the first paragraph of this proof; by (2.7) and (2.8), (u, π) is a classical solution of the NSE for $t \in (0, T)$. The asserted analyticity properties of u then follow immediately, while the inequality (2.6) follows from Fatou's lemma applied to (2.15). ▮

Remark 2.5. Note that the above proof assures that $u + iv \in C^\infty(\mathscr{D})$ and $\pi + i\rho \in C^\infty(\mathscr{D})$.

3. NAVIER–STOKES EQUATIONS WITH PERIODIC BOUNDARY CONDITIONS

The case of periodic boundary conditions requires only minor modifications. Namely, consider the NSE (2.1) with the initial condition (2.2). We assume that u_0, u, π, and f are Ω-periodic where $\Omega = [0, 1]^D (D \geqslant 2)$. We also assume that u_0 is divergence-free and $u_0 \in L^p(\Omega)$ where $p \in (D, \infty)$. For the properties of solutions of the NSE, cf. [CF].

As before, we assume that, for all $t \geqslant 0$, $f(\cdot, t)$ is divergence-free and real-analytic in the space variable with the analyticity radius at least $\delta_f > 0$. If $f + ig$ is its analytic extension, we require (for simplicity) (2.3) and

$$\gamma = \sup_{t \geqslant 0} \sup_{|y| < \delta_f} (\|f(\cdot, y, t)\|_{L^p} + \|g(\cdot, y, t)\|_{L^p}) < \infty,$$

where $\|\cdot\|_{L^p} = \|\cdot\|_{L^p(\Omega)}$.

THEOREM 3.1. *Assume* $\|u_0\|_{L^p} \leqslant M_p < \infty$, *and let*

$$T = \min\left\{\frac{1}{Cp^2 M_p^{2p/(p-D)}}, \frac{1}{Cp^2 M_p^2}, \frac{M_p}{C\gamma}\right\}. \tag{3.1}$$

Then there exists a solution $u \in C([0, T), L^p_{\text{per}}(\mathbb{R}^D)^D)$ *of the NSE with the following property: For every* $t \in (0, T)$, *u is a restriction of an analytic function* $u(x, y, t) + iv(x, y, t)$ *in the region*

$$\mathcal{D}_t = \{(x, y) \in \mathbb{C}^D : |y| \leqslant C^{-1} t^{1/2}, |y| < \delta_f\};$$

moreover,

$$\|u(\cdot, y, t)\|_{L^p} + \|v(\cdot, y, t)\|_{L^p} \leqslant CM_p$$

for $t \in (0, T)$ *and* $(x, y) \in \mathcal{D}_t$.

The only substantial modification in the proof is to establish the periodic analog of Lemma 2.2. A fundamental solution of the heat equation with periodic boundary conditions is

$$\Gamma_p(x, t) = \sum_{k \in \mathbb{Z}^D} \Gamma(x - k, t), \qquad x \in \Omega, \qquad t > 0,$$

where $\Gamma(x, t) = (4\pi t)^{-D/2} \exp(-|x|^2/4t)$.

LEMMA 3.2. (i) *We have*

$$\Gamma_p(x, t) \leqslant \frac{C}{t^{D/2}} \exp\left(-\frac{|x-k|^2}{4t}\right), \qquad x \in \Omega, \qquad 0 < t \leqslant 1$$

and

$$\Gamma_p(x, t) \leqslant C, \qquad x \in \Omega, \qquad t \geqslant 1.$$

(ii) *Also*

$$|\nabla \Gamma_p(x, t)| \leqslant \frac{C}{t^{(D+1)/2}} \exp\left(-\frac{|x|^2}{8t}\right), \qquad x \in \Omega, \qquad 0 < t \leqslant 1$$

and

$$|\nabla \Gamma_p(x, t)| \leqslant \frac{C}{t^{1/2}}, \qquad x \in \Omega, \qquad t \geqslant 1.$$

The proof follows the one from [EK].

Proof. (i) Let $x \in \Omega$ and $t > 0$. Then

$$\begin{aligned}
\Gamma_p(x,t) &= \frac{1}{(4\pi t)^{D/2}} \sum_{k\in\mathbb{Z}^D} \exp\left(-\frac{|x-k|^2}{4t}\right)\\
&\leqslant \frac{1}{(4\pi t)^{D/2}} \sum_{k\in\mathbb{Z}^D} \exp\left(-\frac{|x|^2}{4t}-\frac{|k|^2}{4t}+\frac{|k|}{4t}\right)\\
&\leqslant \frac{1}{(4\pi t)^{D/2}} \exp\left(-\frac{|x|^2}{4t}\right)\left(D+1+\sum_{k\in\mathbb{Z}^D,\ |k|\geqslant\sqrt{2}} \exp\left(-\frac{(2-\sqrt{2})\,|k|^2}{8t}\right)\right)\\
&\leqslant \frac{C}{t^{D/2}} \exp\left(-\frac{|x|^2}{4t}\right)\left(1+\int_{\mathbb{R}^D} \exp\left(-\frac{|w|^2}{Ct}\right)dw\right)\\
&\leqslant \frac{C(1+t^{D/2})}{t^{D/2}} \exp\left(-\frac{|x|^2}{4t}\right).
\end{aligned}$$

(ii) For every $j \in \{1, ..., D\}$, $x \in \Omega$, and $t > 0$, we have

$$\partial_j \Gamma_p(x,t) = -\frac{1}{(4\pi t)^{D/2}} \sum_{k\in\mathbb{Z}^D} \frac{x_j-k_j}{2t} \exp\left(-\frac{|x-k|^2}{4t}\right).$$

Using $se^{-s} \leqslant e^{-s/2}$ for $s \geqslant 0$, we get

$$|\nabla\Gamma_p(x,t)| \leqslant \frac{C}{t^{(D+1)/2}} \exp\left(\frac{-|x|^2}{8t}\right) + \frac{C}{t^{(D+1)/2}} \sum_{k\in\mathbb{Z}^D,\ |k|\geqslant\sqrt{2}} \exp\left(\frac{-|x-k|^2}{8t}\right)$$

and we may proceed as before. ∎

LEMMA 3.3. *We have* $\|\Gamma_p(\cdot,t)\|_{L^1} \leqslant C$ *for* $t > 0$ *and*

$$\|\nabla\Gamma_p\|_{L^{q,1}(S_T)} \leqslant C(q)(T^{(q+D-Dq)/2q} + T^{1/2}), \qquad 1 \leqslant q < \frac{D}{D-1},$$

where $S_T = \Omega \times [0, T]$ *with* $T > 0$.

Proof. This follows immediately from Lemma 3.2. ∎

Now, we form the sequence of approximating solutions $u^{(n)}$ and $\pi^{(n)}$ as before, except that we require additionally

$$\int_\Omega \pi^{(n)}(x)\,dx = 0, \qquad n \in \mathbb{N}.$$

We obtain the equations (2.7) and (2.8), while we may also assume

$$\int_\Omega \pi^{(n)}(x, y, t)\, dx = \int_\Omega \rho^{(n)}(x, y, t)\, dx = 0 \tag{3.2}$$

provided $|y| < \delta_f$ and $t > 0$—this may be verified easily using the Fourier expansions.

In order to estimate the L^p norm of the pressure we use the following well-known estimate.

LEMMA 3.4. *Let* $q \in [2, \infty)$. *Then*

$$\|\pi^{(n)}\|_{L^q(\Omega)} \leqslant Cq(\|u^{(n)}\|^2_{L^{2q}(\Omega)} + \|v^{(n)}\|^2_{L^{2q}(\Omega)})$$
$$\|\rho^{(n)}\|_{L^q(\Omega)} \leqslant Cq(\|u^{(n)}\|^2_{L^{2q}(\Omega)} + \|v^{(n)}\|^2_{L^{2q}(\Omega)}).$$

The proof of Lemma 3.4 is omitted since it is parallel to the estimates on the pressure from [CKN, Section 2C]. (The normalizations (3.2) are needed in the proof.)

Introducing (2.9), (2.10), and (2.11), we obtain identities (2.13) and (2.14) with Γ_p instead of Γ and integrals being taken over Ω instead of $\mathbb{R}^D$. We get

$$\begin{aligned} &\|U^{(n)}\|_{L^{p,\infty}(S_T)} + \|V^{(n)}\|_{L^{p,\infty}(S_T)} \\ &\quad \leqslant C\,\|u_0\|_{L^p} + Cp(T^{1/2 - D/2 + D(p-1)/2p} + T^{1/2}) \\ &\qquad \times (\|U^{(n-1)}\|_{L^{p,\infty}(S_T)} + \|V^{(n-1)}\|_{L^{p,\infty}(S_T)})^2 \\ &\qquad + CT(\|F\|_{L^{p,\infty}(S_T)} + \|G\|_{L^{p,\infty}(S_T)}). \end{aligned}$$

Therefore, (2.15) holds provided

$$0 < T \leqslant \min\left\{\frac{1}{C(pM_p)^{2p/(p-D)}}, \frac{1}{C(pM_p)^2}, \frac{M_p}{C(\|F\|_{L^{p,\infty}(S_T)} + \|G\|_{L^{p,\infty}(S_T)})}\right\}.$$

The following is the analog of Lemma 2.3.

LEMMA 3.5. *Assume*

$$|y| < \min\{\delta_f, C^{-1}T^{1/2}\}$$

and (3.1). *Then*

$$\|u^{(n)}(\cdot, y, t)\|_{L^p} + \|v^{(n)}(\cdot, y, t)\|_{L^p} \leqslant CM_p$$

for all $n \in \mathbb{N}$ *and* $t > 0$.

Proof. The statement follows immediately from the above considerations. ∎

Proof of Theorem 3.1. The proof is identical to the proof of Theorem 2.1, and it is thus omitted. ∎

Now, assume

$$\|u_0\|_{L^\infty} \leqslant M_\infty < \infty$$

and let γ be as above with $p = \infty$. Also, denote

$$\log_+(x) = \max\{\log x, 0\}, \qquad x > 0.$$

THEOREM 3.6. *Let*

$$T = \min\left\{\frac{1}{CM_\infty^2(1+\log_+ M_\infty)^2}, \frac{M_\infty}{C\gamma}\right\}.$$

Then there exists a solution $u \in \bigcap_{p\in(D,\infty)} C([0,T), L^p_{\mathrm{per}}(\mathbb{R}^D)^D)$ *of the NSE with the following property: For every* $t \in (0,T)$, u *is a restriction of an analytic function* $u(x,y,t) + iv(x,y,t)$ *in the region*

$$\mathcal{D}_t = \{(x,y) \in \mathbb{C}^D : |y| \leqslant C^{-1}t^{1/2}, |y| < \delta_f\}.$$

Moreover,

$$\|u(\cdot,y,t)\|_{L^\infty} + \|v(\cdot,y,t)\|_{L^\infty} \leqslant CM_\infty$$

for $t \in (0,T)$ *and* $(x,y) \in \mathcal{D}_t$.

Proof. Note that $\|u_0\|_{L^p} \leqslant M_\infty < \infty$ for every $p \in (D,\infty)$. If $M_\infty \geqslant 2$, we apply Theorem 3.1 with $p = D + \log M_\infty$, while if $M_\infty \leqslant 2$, we use $p = 2D$. ∎

In the case $D = 3$ and $f = 0$, the quantities in the previous theorem may be expressed in terms of the Reynolds number. Namely, consider

$$\begin{aligned} \frac{\partial u}{\partial t} - \nu\,\Delta u + (u\cdot\nabla)\,u + \nabla\pi &= 0 \\ \nabla\cdot u &= 0 \end{aligned} \tag{3.3}$$

in the periodic domain $\Omega = [0,L]^3$, where $\nu, L > 0$; the initial condition is

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^3, \tag{3.4}$$

where u_0 is Ω-periodic, bounded, and divergence-free. Denote by

$$R = \frac{L}{\nu} \| u_0 \|_{L^\infty}$$

the Reynolds number.

COROLLARY 3.7. *Let*

$$T = \frac{L^2}{C\nu R^2 (1 + \log_+ R)^2}$$

Then there exists a solution $u \in \bigcap_{p \in (D, \infty)} C([0, T), L^p_{\mathrm{per}}(\mathbb{R}^3)^3)$ *of* (3.3), (3.4) *with the following property: For every* $t \in (0, T)$, u *is a restriction of an analytic function* $u(x, y, t) + iv(x, y, t)$ *in the region*

$$\mathcal{D}_t = \{(x, y) \in \mathbb{C}^3 : |y| \leqslant C^{-1} (\nu t)^{1/2}\}.$$

Moreover,

$$\|u(\cdot, y, t)\|_{L^\infty} + \|v(\cdot, y, t)\|_{L^\infty} \leqslant C \frac{\nu R}{L}$$

for $t \in (0, T)$ *and* $(x, y) \in \mathcal{D}_t$.

In particular, the analyticity radius of the solution u at time

$$t_0 = \frac{L^2}{C\nu R^2 (1 + \log_+ R)^2}$$

is greater than or equal to

$$\delta_0 = \frac{L}{CR(1 + \log_+ R)}.$$

Proof. Corollary 3.7 can be reduced to Theorem 3.6 by introducing

$$\tilde{u}(x, t) = \frac{L}{\nu} u\left(Lx, \frac{L^2 t}{\nu}\right)$$

and

$$\tilde{\pi}(x, t) = \frac{L^2}{\nu^2} u\left(Lx, \frac{L^2 t}{\nu}\right). \quad \blacksquare$$

4. ANALYTICITY IN L^∞ FOR THE KURAMOTO–SIVASHINSKY EQUATION

Consider the Kuramoto–Sivashinsky equation (KSE)

$$u_t + u_{xxxx} + u_{xx} + uu_x = 0 \tag{4.1}$$

with the initial condition

$$u(x, 0) = u_0(x), \tag{4.2}$$

where $u_0 \in L^\infty(\mathbb{R})$. A function u: $\mathbb{R} \times [0, T) \to \mathbb{R}$, where $T > 0$, is a solution of the initial value problem (4.1), (4.2) if it is a classical solution of (4.1) for $t \in (0, T)$, if $u, u_x \in L^\infty(\mathbb{R} \times I)$ for every compact interval $I \subseteq (0, T)$, and if (4.2) is satisfied in the following sense:

$$\lim_{t \to 0^+} \|u(\cdot, t) - u_{\text{lin}}(\cdot, t)\|_{L^\infty(\mathbb{R})} = 0,$$

where

$$u_{\text{lin}}(x, t) = \int \Gamma(x - y, t)\, u_0(y)\, dy$$

and

$$\Gamma(x, t) = \frac{1}{2\pi} \int e^{-\xi^4 t + i\xi x}\, d\xi$$

is a fundamental solution of the equation $v_t + v_{xxxx} = 0$.

THEOREM 4.1. *There exists a universal constant $C > 0$ such that for every $u_0 \in L^\infty(\mathbb{R})$ there exists a solution u of the KSE, where $T = C^{-1}(1 + \|u_0\|_{L^\infty(\mathbb{R})}^{4/3})^{-1}$, with the following property: For every $t \in (0, T)$, $u(x, t)$ is a restriction of an analytic function $u(x, y, t) + iv(x, y, t)$ in the domain*

$$\mathscr{D}_t = \{(x, y) \in \mathbb{C} : C|y| \leqslant t^{1/4}\}.$$

Moreover,

$$|u(x, y, t) + iv(x, y, t)| \leqslant C\, \|u_0\|_{L^\infty(\mathbb{R})}, \qquad (x, y) \in \mathscr{D}_t$$

for $t \in (0, T)$.

In order to solve this equation, we form a sequence of approximating solutions obtained in the following way. Let $u^{(0)}=0$. We obtain $u^{(n)}(n\in\mathbb{N})$ by solving

$$u_t^{(n)}+u_{xxxx}^{(n)}=-u_{xx}^{(n-1)}-u^{(n-1)}u_x^{(n-1)}$$
$$u^{(n)}(x,0)=u_0(x).$$

For every $n\in\mathbb{N}_0$, $u^{(n)}$ is a classical solution for $t>0$, and, for every fixed $t>0$, $u^{(n)}$ may be extended to an entire function, denoted by

$$u^{(n)}(x,y,t)+iv^{(n)}(x,y,t).$$

In order to pass to a limit, we proceed to obtain n-independent bounds on $u^{(n)}$ and $v^{(n)}$. Now, for every $\alpha\in\mathbb{R}$, consider the functions

$$U_\alpha^{(n)}(x,t)=u^{(n)}(x,\alpha t,t)$$
$$V_\alpha^{(n)}(x,t)=v^{(n)}(x,\alpha t,t)$$

for $n\in\mathbb{N}$, $x\in\mathbb{R}$, and $t\geqslant 0$. These functions (for $n\in\mathbb{N}$) satisfy the system

$$\begin{aligned}U_t^{(n)}+U_{xxxx}^{(n)}&=-\alpha V_x^{(n)}-U_{xx}^{(n-1)}-U^{(n-1)}U_x^{(n-1)}+V^{(n-1)}V_x^{(n-1)}\\ V_t^{(n)}+V_{xxxx}^{(n)}&=\alpha U_x^{(n)}-V_{xx}^{(n-1)}-U^{(n-1)}V_x^{(n-1)}-V^{(n-1)}U_x^{(n-1)}\end{aligned}\tag{4.3}$$

the subscript α being omitted for simplicity, with initial conditions

$$\begin{aligned}U^{(n)}(x,0)&=u_0(x)\\ V^{(n)}(x,0)&=0.\end{aligned}\tag{4.4}$$

Note that $U_0^{(n)}(x,t)=u^{(n)}(x,t)$ and $V_0^{(n)}(x,t)=0$ for all $x\in\mathbb{R}$ and $t\geqslant 0$. The equations (4.3) and (4.4) may be rewritten in the integral form as

$$\begin{aligned}U^{(n)}(x,t)=&\int\Gamma(x-y,t)\,u_0(y)\,dy-\int_0^t\int\partial_{xx}\Gamma(x-y,t-s)\,U^{(n-1)}(y,s)\,dy\,ds\\ &-\frac{1}{2}\int_0^t\int\partial_x\Gamma(x-y,t-s)\,U^{(n-1)}(y,s)^2\,dy\,ds\\ &+\frac{1}{2}\int_0^t\int\partial_x\Gamma(x-y,t-s)\,V^{(n-1)}(y,s)^2\,dy\,ds\\ &-\alpha\int_0^t\int\partial_x\Gamma(x-y,t-s)\,V^{(n)}(y,s)\,dy\,ds\end{aligned}\tag{4.5}$$

and

$$V^{(n)}(x,t) = -\int_0^t \int \partial_{xx}\Gamma(x-y,t-s)\, V^{(n-1)}(y,s)\,dy\,ds$$
$$-\int_0^t \int \partial_x\Gamma(x-y,t-s)\, U^{(n-1)}(y,s)\, V^{(n-1)}(y,s)\,dy\,ds$$
$$+\alpha\int_0^t \int \partial_x\Gamma(x-y,t-s)\, U^{(n)}(y,s)\,dy\,ds. \tag{4.6}$$

Due to the scaling

$$\Gamma(x,t) = \frac{1}{t^{1/4}}\,\Gamma\left(\frac{x}{t^{1/4}},1\right), \qquad x\in\mathbb{R}, \qquad t>0$$

we have the following estimates.

LEMMA 4.2. *There exists a universal constant C such that*

$$\|\Gamma(\cdot,t)\|_{L^1(\mathbb{R})} \leqslant C, \qquad t>0$$

and

$$\|\partial_x^n \Gamma\|_{L^1(S_T)} \leqslant CT^{(4-1)/4}, \qquad n\in\{0,1,2,3\},$$

where $S_T=\mathbb{R}\times[0,T]$ with $T>0$.

The proof is straightforward and is thus omitted.

Let $T>0$ be arbitrary. By Lemma 4.2, we obtain from (4.5)

$$\|U^{(n)}\|_{L^\infty(S_T)} \leqslant C\,\|u_0\|_{L^\infty(\mathbb{R})} + CT^{1/2}\,\|U^{(n-1)}\|_{L^\infty(S_T)} + CT^{3/4}\,\|U^{(n-1)}\|^2_{L^\infty(S_T)}$$
$$+ CT^{3/4}\,\|V^{(n-1)}\|^2_{L^\infty(S_T)} + C\,|\alpha|\,T^{3/4}\,\|V^{(n)}\|_{L^\infty(S_T)}$$

while we get from (4.6)

$$\|V^{(n)}\|_{L^\infty(S_T)} \leqslant CT^{1/2}\,\|V^{(n-1)}\|_{L^\infty(S_T)} + CT^{3/4}$$
$$\times\|U^{(n-1)}\|_{L^\infty(S_T)}\,\|V^{(n-1)}\|_{L^\infty(S_T)}$$
$$+ C\,|\alpha|\,T^{3/4}\,\|U^{(n)}\|_{L^\infty(S_T)}.$$

Assuming $T\leqslant C^{-1}\,|\alpha|^{-4/3}$ for large enough C, we get

$$\|U^{(n)}\|_{L^\infty(S_T)} + \|V^{(n)}\|_{L^\infty(S_T)}$$
$$\leqslant C\,\|u_0\|_{L^\infty(\mathbb{R})} + CT^{1/2}(\|U^{(n-1)}\|_{L^\infty(S_T)} + \|V^{(n-1)}\|_{L^\infty(S_T)})$$
$$+ CT^{3/4}(\|U^{(n-1)}\|_{L^\infty(S_T)} + \|V^{(n-1)}\|_{L^\infty(S_T)})^2.$$

From these inequalities, we deduce by induction

$$\|U^{(n)}\|_{L^\infty(S_T)} + \|V^{(n)}\|_{L^\infty(S_T)} \leqslant C\|u_0\|_{L^\infty(\mathbb{R})}, \qquad n \in \mathbb{N}$$

provided

$$0 \leqslant T \leqslant \frac{1}{C} \min\left\{1, \frac{1}{|\alpha|^{4/3}}, \frac{1}{\|u_0\|_{L^\infty(\mathbb{R})}^{4/3}}\right\}.$$

Proof of Theorem 4.1 The analogs of (2.18) and (2.19) are

$$\begin{aligned}\|U^{(n)} - u^{(n-1)}\|_{L^\infty(S_T)} \leqslant{} & CT^{1/2}\, \|u^{(n-1)} - u^{(n-2)}\|_{L^\infty(S_T)} \\ & + CT^{3/4}(\|u^{(n-1)}\|_{L^\infty(S_T)} + \|u^{(n-2)}\|_{L^\infty(S_T)}) \\ & \times \|u^{(n-1)} - u^{(n-2)}\|_{L^\infty(S_T)}\end{aligned}$$

and

$$\|u^{(n)} - u_{\mathrm{lin}}\|_{L^\infty(S_T)} \leqslant CT^{1/2}\, \|u^{(n-1)}\|_{L^\infty(S_T)} + CT^{3/4}\, \|u^{(n-1)}\|_{L^\infty(S_T)}^2.$$

The rest follows as in the proof of Theorem 2.1. ∎

Remark 4.3. By rescaling, we can also treat the equation

$$u_t + au_{xxxx} + bu_{xx} + cuu_x = 0$$
$$u(x, 0) = u_0(x),$$

where $a, b > 0$, $c \in \mathbb{R}$, and $u_0 \in L^\infty(\mathbb{R})$. Namely,

$$\tilde{u}(x, t) = \frac{ca^{1/2}}{b^{3/2}}\, u\left(\frac{a^{1/2}x}{b^{1/2}}, \frac{at}{b^2}\right), \qquad x \in \mathbb{R}, \qquad t \geqslant 0$$

satisfies the KSE. Therefore, we obtain that for

$$0 < t \leqslant \frac{a}{C(b^{3/2} + |c|\, a^{1/2}\, \|u_0\|_{L^\infty})^{4/3}}$$

the space-analyticity radius of u is at least $C^{-1}a^{1/4}t^{1/4}$. The same result also follows by repeating the proof of Theorem 4.1. ∎

Now, we restrict ourselves to the initial data u_0 which are L-periodic, where $L > 0$ is fixed, and for which $\int_0^L u_0 = 0$. Then KSE possesses the global attractor $\mathscr{A}$ (cf. [CEES1]). Also, the quantity

$$M_\infty(L) = \sup_{u_0 \in \mathscr{A}} \|u_0\|_{L^\infty(\mathbb{R})}$$

is finite.

THEOREM 4.4. *Let* $u_0 \in \mathscr{A}$. *Then* u_0 *is real-analytic with the real-analyticity radius at least* $1/C(M_\infty + 1)^{1/3}$.

Proof. The assertion follows immediately from Theorem 4.1. ∎

It was conjectured in [CEES2] that the real-analyticity radius of functions $u_0 \in \mathscr{A}$ can be estimated from below by a positive constant independent of L. From Theorem 4.4, it follows that this holds provided $M_\infty(L)$ can be bounded by a quantity independent of L. Numerical evidence indeed suggests this ([E]), but the fact is apparently open. A similar fact holds for the complex Ginzburg–Landau equation ([C], [CE]); however, the method does not apply to the KSE.

Similar, but less precise, analyticity result was proven in [TBDVT].

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