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Space Analyticity for the Nonlinear Heat Equation in a Bounded Domain

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1. INTRODUCTION

In [FoT], Foias and Temam introduced a method for estimating the space-analyticity radius of solutions of the Navier–Stokes equation with periodic boundary conditions. For the numerous applications, see e.g. [CEES, CRT, DTH, FT, Gr, K, LO]. In our previous paper [GK], we did not use the Fourier series and were thus able to treat the Navier–Stokes and other semilinear parabolic equations with singular (L^p) initial data. In particular, we expressed the analyticity radius in terms of the Reynolds number.

Although it is well-known (cf. [M1, M2, G, HKR]) that solutions are analytic in the space variable also in the case of Dirichlet boundary conditions, it was not clear how to generalize the method of Foias and Temam to treat this case as well. The purpose of this paper is to show that this is indeed possible and illustrate the method on the non-linear heat equation. The main idea is to establish an energy inequality for the quantity

$$\int (u(x, \alpha\psi(x)\, t, t)^2 + v(x, \alpha\psi(x)\, t, t)^2)^p \, dx,$$

where ψ is a test function, p is suitably large, and $u + iv$ is the analytic extension of the solution u.

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The paper is organized as follows. Theorem 2.1 (see also Remark 2.2) contains the main result. The rest of Section 2 contains the proof and a corollary addressing bounded initial data.

For some interesting methods for establishing analyticity of solutions of evolution equations, also see [B, BB, KM, TBDVT]. In particular, [BB] contains an explicit estimate of the analyticity domain as a function of the initial data and the distance from the boundary for analytic solutions of the Euler equation.

2. THE MAIN RESULT

Consider the nonlinear heat equation

$$\partial_t u - \Delta u = u^k, \qquad t > 0 \tag{2.1}$$

$$u|_{\partial\Omega} = 0, \qquad t > 0 \tag{2.2}$$

$$u(x, 0) = u_0(x), \qquad x \in \Omega \tag{2.3}$$

where $k \in \{2, 3, ...\}$. We assume that $\Omega \subseteq \mathbb{R}^D$ $(D \in \mathbb{N})$ is a bounded domain with a smooth boundary. We also assume that $u_0 \in L^{2p}(\Omega)$ with $2p \in [1, \infty)$.

A function $u: \bar{\Omega} \times [0, T) \to \mathbb{R}$, where $T > 0$, is a solution of the initial value problem (2.1)–(2.3) if $u \in C([0, T), L^{2p}(\Omega))$ with $u(0) = u_0$, and if u satisfies (2.1) and (2.2) in the classical sense for $t \in (0, T)$.

In order to solve (2.1)–(2.3), we form a sequence of approximations $\{u^{(n)}\}_{n=0}^{\infty}$ in the following way: Set $u^{(0)} = 0$, and then, for each $n \in \mathbb{N}$, let $u^{(n)}$ be a solution of

$$\partial_t u^{(n)} - \Delta u^{(n)} = (u^{(n-1)})^k, \qquad (x, t) \in \Omega \times (0, \infty)$$

$$u^{(n)}(x, 0) = u_0(x), \qquad x \in \Omega$$

with Dirichlet boundary conditions. If $2p > D(k-1)/2$, the sequence $u^{(n)}$ converges in a certain topology to a unique solution $u \in C([0, T), L^{2p}(\Omega))$ of (2.1) for some $T > 0$ depending on Ω, k, and $\|u_0\|_{L^{2p}}$ ([BC, W]). Certain critical and subcritical $2p$ were also considered in [BC].

It is well-known that the solution u is analytic in the space variable at every $x_0 \in \Omega$ and $t \in (0, T)$. Our main theorem, which is stated next, provides an estimate of the analyticity radius.

Let $2p \geqslant k$ if $D = 1$, $2p > k$ if $D = 2$, and $2p \geqslant (Dk - D + 2)/2$ if $D \geqslant 3$.

THEOREM 2.1. *Denote*

$$M_{2p}=\left(\int_{\Omega}|u_0(x)|^{2p}\right)^{1/2p}.$$

Let $x_0\in\Omega$ *and* $d_0=\operatorname{dist}(x_0,\partial\Omega)$. *If* $0<t\leqslant d_0^2/C$ *and*

$$t\leqslant\frac{1}{Cp}\left(1+M_{2p}^{-4p(k-1)/(4p-Dk+D)}\right) \tag{2.4}$$

the space analyticity radius $\delta_{x_0}(t)$ *of* u *at point* x_0 *and time* t *satisfies*

$$\delta_{x_0}(t)\geqslant\frac{1}{Cp^{1/2}}\min\{t^{1/2},d_0\}.$$

The symbol C above and in the sequel denotes a positive constant which may depend only on Ω, D, and k but not on any other quantity. It may, however, also depend on p, but only in the case $D=2$ and as $2p\to k$.

It is worth pointing out that the exponent in (2.4) is the same as if we considered the nonlinear heat equation with no boundary conditions and followed the method in [GK].

Remark 2.2. Theorem 2.1 requires only minor modifications if we substitute u^k in (2.1) with $p_k(u)$ where p_k is a polynomial of degree k. Namely, if $p_k(0)=0$, the statement of Theorem 2.1 is identical, while if $p_k(0)\neq 0$, we need to substitute (2.4) with

$$0<t\leqslant\frac{1}{Cp}\left(1+\max\{1,M_{2p}^{-4p(k-1)/(4p-Dk+D)}\}\right).$$

The changes in the proof are straightforward.

In the proof, the following statement is needed.

LEMMA 2.3. *Let* $T>0$ *and* $u_0\in C(\Omega)$. *Assume that* $f\in C^{\infty}(\bar{\Omega}\times[0,T))$ *admits an extension*

$$f(x,y,t)+ig(x,y,t)\in C^{\infty}(\mathscr{D}),$$

where $\mathscr{D}=\{(x,y,t)\in\mathbb{C}^D\times(0,T): x\in\Omega,\quad |y|<\operatorname{dist}(x,\partial\Omega)\}$ *such that* $f(x,y,t_0)+ig(x,y,t_0)$ *is analytic in the domain* $\mathscr{D}_{t_0}=\{(x,y,t_0)\in\mathscr{D}: t=t_0\}$ *for every* $t_0\in(0,T)$. *Then the solution of the problem*

$$\begin{aligned} \partial_t u - \Delta u &= f, && t > 0 \\ u|_{\partial\Omega} &= 0, && t > 0 \\ u(x, 0) &= u_0(x), && x \in \Omega \end{aligned}$$

admits an extension

$$u(x, y, t) + iv(x, y, t) \in C^\infty(\mathscr{D})$$

such that $u(x, y, t_0) + iv(x, y, t_0)$ *is analytic in* $\mathscr{D}_{t_0}$ *for every* $t_0 \in (0, T)$. *Moreover,*

$$\begin{aligned} \partial_t u - \Delta u &= f \\ \partial_t v - \Delta v &= g \end{aligned}$$

for $(x, y, t) \in \mathscr{D}$.

The proof for the homogeneous case $f = 0$ is given in [J, pp. 218–219]; the changes for the non-homogeneous case are straight-forward, and the details are thus omitted.

Proof of Theorem 2.1. By Lemma 2.3, $u^{(n)}$ is analytic in x for every $t > 0$ with the analyticity radius at least $d(x) = \operatorname{dist}(x, \partial\Omega)$. The analytic extension $u^{(n)}(x, y, t) + iv^{(n)}(x, y, t)$, where $x \in \Omega$, $|y| < d(x)$, and $t > 0$, satisfies

$$\begin{aligned} \partial_t u^{(n)} - \Delta u^{(n)} &= g(u^{(n-1)}, v^{(n-1)}) \\ \partial_t v^{(n)} - \Delta v^{(n)} &= h(u^{(n-1)}, v^{(n-1)}), \end{aligned}$$

where

$$\begin{aligned} g(x, y) &= \sum_{j=0}^{[k/2]} \binom{k}{2j} (-1)^j x^{k-2j} y^{2j} \\ h(x, y) &= \sum_{j=0}^{[(k-1)/2]} \binom{k}{2j+1} (-1)^j x^{k-2j-1} y^{2j+1}. \end{aligned}$$

Additionally, we have the Cauchy–Riemann equations

$$\begin{aligned} \frac{\partial u^{(n)}}{\partial x_j} &= \frac{\partial v^{(n)}}{\partial y_j} \\ \frac{\partial u^{(n)}}{\partial y_j} &= -\frac{\partial v^{(n)}}{\partial x_j} \end{aligned} \tag{2.5}$$

for $j = 1, ..., D$.

Now, fix $x_0 \in \Omega$, and let $d_0 = \operatorname{dist}(x_0, \partial\Omega)$. Choose $\psi \in C_0^\infty(\Omega)$ such that $0 \leqslant \psi(x) \leqslant 1$ for $x \in \Omega$,

$$\operatorname{supp} \psi \subseteq B(x_0, d_0/2)$$

and

$$\psi(x) = 1, \qquad x \in B(x_0, d_0/4). \tag{2.6}$$

Moreover, we may assume

$$|\nabla^j \psi(x)| \leqslant \frac{C}{d_0^j}, \qquad x \in \Omega$$

for $j \in \{1, 2\}$.

For $\alpha \in \mathbb{R}^D$, consider the functions

$$\begin{aligned} U_\alpha^{(n)}(x, t) &= u^{(n)}(x, \alpha t \psi(x), t) \\ V_\alpha^{(n)}(x, t) &= v^{(n)}(x, \alpha t \psi(x), t) \end{aligned} \tag{2.7}$$

for $x \in \Omega$ and $t \geqslant 0$, where we assume

$$|\alpha|\ t \leqslant \frac{d_0}{C}$$

in order to assure that $(x, \alpha t \psi(x), t)$ belongs to the domain of analyticity of $u^{(n)} + iv^{(n)}$. Note that

$$U_\alpha^{(n)}(x, t) = u(x, t)$$

$$V_\alpha^{(n)}(x, t) = v(x, t)$$

for $x \in \Omega \backslash \operatorname{supp} \psi$. We proceed to find the equations for $U_\alpha^{(n)}(x, t)$ and $V_\alpha^{(n)}(x, t)$.

Denoting $\partial_j = \partial/\partial x_j$ and $\partial_j' = \partial/\partial y_j$, we get by differentiating (2.7)

$$\partial_j U^{(n)} = \partial_j u^{(n)} + \alpha_l t\, \partial_j \psi\, \partial_l' u^{(n)} = \partial_j u^{(n)} - \alpha_l t\, \partial_j \psi\, \partial_l v^{(n)} \tag{2.8}$$

and

$$\partial_j V^{(n)} = \partial_j v^{(n)} + \alpha_l t \partial_j \psi\, \partial_l' v^{(n)} = \partial_j v^{(n)} + \alpha_l t\, \partial_j \psi\, \partial_l u^{(n)} \tag{2.9}$$

where we used the Cauchy–Riemann equations (2.5) and where we omitted the subscript α for simplicity. Now, if $|\alpha|\, t\, |\nabla\psi|$ is suitably small, i.e., if

$$|\alpha| t \leqslant \frac{d_0}{C}$$

for suitably large $C = C(D) > 0$, we may solve (2.8) and (2.9) for $\partial_j u^{(n)}$ and $\partial_j v^{(n)}$ thus obtaining

$$\begin{aligned} \partial_j u^{(n)} &= \tilde{b}^{11}_{jk}\, \partial_k U^{(n)} + \tilde{b}^{12}_{jk}\, \partial_k V^{(n)} \\ \partial_j v^{(n)} &= \tilde{b}^{21}_{jk}\, \partial_k U^{(n)} + \tilde{b}^{22}_{jk}\, \partial_k V^{(n)}, \end{aligned} \tag{2.10}$$

where

$$\| \tilde{b}^{lm}_{jk} \|_{L^\infty} \leqslant C.$$

Differentiating (2.8) and (2.9) and using the Cauchy–Riemann equations, we get

$$\begin{aligned} \partial_{jk} U^{(n)} = {} & \partial_{jk} u^{(n)} - \alpha_l t \partial_k \psi\, \partial_{jl} v^{(n)} - \alpha_l t\, \partial_j \psi\, \partial_{kl} v^{(n)} \\ & - \alpha_l \alpha_m t^2\, \partial_j \psi\, \partial_k \psi\, \partial_{lm} u^{(n)} - \alpha_l t\, \partial_{jk} \psi\, \partial_l v^{(n)} \end{aligned}$$

and

$$\begin{aligned} \partial_{jk} V^{(n)} = {} & \partial_{jk} v^{(n)} + \alpha_l t\, \partial_k \psi\, \partial_{jl} u^{(n)} + \alpha_l t\, \partial_j \psi\, \partial_{kl} u^{(n)} \\ & - \alpha_l \alpha_m t^2\, \partial_j \psi\, \partial_k \psi\, \partial_{lm} v^{(n)} + \alpha_l t\, \partial_{jk} \psi\, \partial_l u^{(n)} . \end{aligned}$$

Assuming

$$|\alpha| t \leqslant \frac{d_0}{C\sqrt{p}} \tag{2.11}$$

for large enough $C = C(D) > 0$, we obtain

$$\begin{aligned} \Delta u^{(n)} = {} & \Delta U^{(n)} + \partial_j (a^{11}_{jk}\, \partial_k U^{(n)}) + \partial_j (a^{12}_{jk}\, \partial_k V^{(n)}) \\ & + \tilde{c}^{11}_j\, \partial_j U^{(n)} + \tilde{c}^{12}_j\, \partial_j V^{(n)} \end{aligned}$$

and

$$\begin{aligned} \Delta v^{(n)} = {} & \Delta V^{(n)} + \partial_j (a^{21}_{jk}\, \partial_k U^{(n)}) + \partial_j (a^{22}_{jk}\, \partial_k V^{(n)}) \\ & + \tilde{c}^{21}_j\, \partial_j U^{(n)} + \tilde{c}^{22}_j\, \partial_j V^{(n)} \end{aligned}$$

where

$$\|a_{jk}^{lm}\|_{L^\infty} \leqslant \frac{1}{C\sqrt{p}} \tag{2.12}$$

and

$$\|\tilde{c}_{jk}^{lm}\|_{L^\infty} \leqslant \frac{C}{d_0\sqrt{p}}.$$

Also, using (2.10),

$$\partial_t U^{(n)} = \partial_t u^{(n)} + \alpha_l \psi\, \partial_l u^{(n)} = \partial_t u^{(n)} + \alpha_l \psi\, \tilde{b}_{lk}^{11}\, \partial_k U^{(n)} + \alpha_l \psi\, \tilde{b}_{lk}^{12}\, \partial_k V^{(n)}$$

and, similarly,

$$\partial_t V^{(n)} = \partial_t v^{(n)} + \alpha_l \psi\, \tilde{b}_{lk}^{21}\, \partial_k U^{(n)} + \alpha_l \psi\, \tilde{b}_{lk}^{22}\, \partial_k V^{(n)}.$$

Therefore, the equation for $U^{(n)}$ is

$$\begin{aligned} \partial_t U^{(n)} = {} & \Delta U^{(n)} + \partial_j (a_{jk}^{11}\, \partial_k U^{(n)}) + \partial_j (a_{jk}^{12}\, \partial_k V^{(n)}) + b_j^{11}\, \partial_j U^{(n)} + b_j^{12}\, \partial_j V^{(n)} \\ & + \alpha_l\, c_{lk}^{11}\, \partial_k U^{(n)} + \alpha_l\, c_{lk}^{12}\, \partial_k V^{(n)} + g(U^{(n-1)}, V^{(n-1)}) \end{aligned} \tag{2.13}$$

while the equation for $V^{(n)}$ is

$$\begin{aligned} \partial_t V^{(n)} = {} & \Delta V^{(n)} + \partial_j (a_{jk}^{21}\, \partial_k U^{(n)}) + \partial_j (a_{jk}^{22}\, \partial_k V^{(n)}) + b_j^{21}\, \partial_j U^{(n)} + b_j^{22}\, \partial_j V^{(n)} \\ & + \alpha_l\, c_{lk}^{21}\, \partial_k U^{(n)} + \alpha_l\, c_{lk}^{22}\, \partial_k V^{(n)} + h(U^{(n-1)}, V^{(n-1)}). \end{aligned} \tag{2.14}$$

The coefficients satisfy

$$\|b_j^{lm}\|_{L^\infty} \leqslant \frac{C}{d_0\sqrt{p}} \tag{2.15}$$

and

$$\|c_{jk}^{lm}\|_{L^\infty} \leqslant C. \tag{2.16}$$

Note that (2.13) and (2.14) reduce to

$$\partial_t U^{(n)} = \Delta U^{(n)} - \alpha_j\, \partial_j V^{(n)} + g(U^{(n-1)}, V^{(n-1)})$$

$$\partial_t V^{(n)} = \Delta V^{(n)} + \alpha_j\, \partial_j U^{(n)} + h(U^{(n-1)}, V^{(n-1)})$$

if $x \in \Omega \backslash \operatorname{supp} \psi$. Now, consider

$$\phi_n(t) = \int_\Omega E_n(x, t)^p \, dx,$$

where $E_n(x, t) = U^{(n)}(x, t)^2 + V^{(n)}(x, t)^2 + \delta$ and $\delta > 0$. (Later we will let $\delta \to 0$.)

Differentiating, we obtain

$$\frac{1}{2p} \phi_n'(t) = \int_\Omega E_n^{p-1} (U^{(n)} \, \partial_t U^{(n)} + V^{(n)} \, \partial_t V^{(n)}) = I_1 + I_2 + I_3 + I_4 + I_5,$$

where the expressions I_1–I_5 are given below. The first term is

$$\begin{aligned} I_1 &= \int_\Omega E_n^{p-1} (U^{(n)} \, \Delta U^{(n)} + V^{(n)} \, \Delta V^{(n)}) \\ &= -\int_\Omega E_n^{p-1} (|\nabla U^{(n)}|^2 + |\nabla V^{(n)}|^2) \\ &\quad - 2(p-1) \int_\Omega E_n^{p-2} (U^{(n)} \, \partial_j U^{(n)} + V^{(n)} \, \partial_j V^{(n)}) \\ &\quad \times (U^{(n)} \, \partial_j U^{(n)} + V^{(n)} \, \partial_j V^{(n)}). \end{aligned} \tag{2.17}$$

The second term may be estimates as

$$\begin{aligned} I_2 &= \int_\Omega E_n^{p-1} \, (\partial_j (a_{jk}^{11} \, \partial_k U^{(n)}) \, U^{(n)} + \partial_j (a_{jk}^{12} \, \partial_k V^{(n)}) \, U^{(n)} \\ &\quad + \partial_j (a_{jk}^{21} \, \partial_k U^{(n)}) \, V^{(n)} + \partial_j (a_{jk}^{22} \, \partial_k V^{(n)}) \, V^{(n)}) \\ &\leqslant \frac{1}{C} \int_\Omega E_n^{p-1} (|\nabla U^{(n)}|^2 + |\nabla V^{(n)}|^2) \\ &\quad + \frac{p}{C} \int_\Omega E_n^{p-2} \, |a_{jk}^{11} \, U^{(n)} \, \partial_k U^{(n)} + a_{jk}^{12} \, U^{(n)} \, \partial_k V^{(n)} \\ &\quad + a_{jk}^{21} \, V^{(n)} \, \partial_k U^{(n)} + a_{jk}^{22} \, V^{(n)} \, \partial_k V^{(n)}| \, |U^{(n)} \, \partial_j U^{(n)} + V^{(n)} \, \partial_j V^{(n)}| \\ &\leqslant \frac{1}{4} |I_1| \end{aligned}$$

where we used (2.12) and (2.17). The third term equals

$$I_3=\int_\Omega E_n^{p-1}(b_j^{11}\,U^{(n)}\,\partial_j U^{(n)}+b_j^{12}\,U^{(n)}\,\partial_j V^{(n)}$$

$$+b_j^{21}\,V^{(n)}\,\partial_j U^{(n)}+b_j^{22}\,V^{(n)}\,\partial_j V^{(n)})$$

$$\leqslant\frac{1}{4}\,|I_1|+\frac{C}{d_0^2 p}\,\phi_n(t)$$

where we used (2.15). Similarly, (2.16) implies

$$I_4=\int_\Omega E_n^{p-1}(\alpha_l\,c_{lk}^{11}\,U^{(n)}\,\partial_k U^{(n)}+\alpha_l\,c_{lk}^{12}\,U^{(n)}\,\partial_k V^{(n)}$$

$$+\alpha_l\,c_{lk}^{21}\,V^{(n)}\,\partial_k U^{(n)}+\alpha_l\,c_{lk}^{22}\,V^{(n)}\,\partial_k V^{(n)})$$

$$\leqslant\frac{1}{4}\,|I_1|+C\,|\alpha|^2\,\phi_n(t).$$

The last term is

$$I_5=\int_\Omega E_n^{p-1}(U^{(n)}g(U^{(n-1)},\,V^{(n-1)})+V^{(n)}h(U^{(n-1)},\,V^{(n-1)}))$$

$$\leqslant C\int_\Omega E_n^{p-1/2}\,E_{n-1}^{k/2}\leqslant C\left(\int_\Omega E_n^{(2p-1)p/(2p-k)}\right)^{(2p-k)/2p}\phi_{n-1}(t)^{k/2p}$$

where we used $2p\geqslant k$. At this point, we employ the Gagliardo–Nirenberg inequality

$$\|A\|_{L^\beta(\Omega)}\leqslant C\,\|A\|_{L^2(\Omega)}^{1+D/\beta-D/2}\,\|\nabla A\|_{L^2(\Omega)}^{D/2-D/\beta},$$

where $2\leqslant\beta\leqslant\infty$ if $D=1$, $2\leqslant\beta<\infty$ if $D=2$, and $2\leqslant\beta\leqslant 2D/(D-2)$ if $D\geqslant 3$. Note that the constant C may depend on β if $D=2$ and $\beta\to\infty$. Now, letting $A=E_n^{p/2}$ and $\beta=2(2p-1)/(2p-k)$, we get

$$\left(\int_\Omega E_n^{(2p-1)p/(2p-k)}\right)^{(2p-k)/2(2p-1)}$$

$$\leqslant C\phi_n(t)^{(4p-Dk+D-2)/4(2p-1)}\left(\int_\Omega E_n^{p-2}(U^{(n)}\,\partial_j U^{(n)}+V^{(n)}\,\partial_j V^{(n)})\right.$$

$$\left.\times(U^{(n)}\,\partial_j U^{(n)}+V^{(n)}\,\partial_j V^{(n)})\right)^{D(k-1)/4(2p-1)}.$$

Raising this inequality to power $(2p-1)/p$ and multiplying the resulting estimate by $\phi_{n-1}(t)^{k/2p}$ leads to

$$\begin{aligned} I_5 &\leqslant C\phi_n(t)^{(4p-Dk+D-2)/4p}\,|I_1|^{D(k-1)/4p}\,\phi_{n-1}(t)^{k/2p} \\ &\leqslant \tfrac{1}{4}\,|I_1|+C\phi_n(t)^{(4p-Dk+D-2)/(4p-Dk+D)}\,\phi_{n-1}(t)^{2k/(4p-Dk+D)}. \end{aligned}$$

Summarizing, we arrive at the inequality

$$\begin{aligned} \frac{1}{2p}\,\phi_n'(t) \leqslant{}& \frac{C}{d_0^2 p}\,\phi_n(t)+C\,|\alpha|^2\phi_n(t) \\ &+C\phi_n(t)^{(4p-Dk+D-2)/(4p-Dk+D)}\,\phi_{n-1}(t)^{2k/(4p-Dk+D)}. \end{aligned}$$

Assume

$$\phi_n(0)=\int_\Omega (u_0(x)^2+\delta)^p = M_{2p}(\delta)^{2p}$$

and note that $M_{2p}(0)=M_{2p}$. By induction, we get

$$\phi_n(t)\leqslant 2M_{2p}(\delta)^{2p}$$

provided

$$0<t\leqslant\frac{1}{Cp}\min\left\{d_0^2 p,\ \frac{1}{|\alpha|^2},\ \frac{1}{M_{2p}(\delta)^{4p(k-1)/(4p-Dk+D)}}\right\}.$$

Also, t is restricted by (2.11).

By sending $\delta\to 0$ and using (2.6), we conclude the following: If

$$0<t\leqslant T=\frac{1}{Cp}\min\left\{d_0^2 p,\ \frac{1}{M_{2p}^{4p(k-1)/(4p-Dk+D)}}\right\}, \tag{2.18}$$

we have

$$\int_{B(x_0,\,d_0/4)} (u^{(n)}(x,y,t)^2+v^{(n)}(x,y,t)^2)^p\,dx\leqslant 2M_{2p}^{2p}$$

provided

$$|y|<\frac{1}{Cp^{1/2}}\min\{t^{1/2},d_0\}.$$

It is easy to check that if C in (2.18) is suitably large, the functions $u^{(n)}(x,t)$ converge uniformly in $C([0,T],L^{2p}(\Omega))$ to a solution $u\in C([0,T),$ $L^{2p}(\Omega))$ [BC]. It is also easy to verify (cf. [GK]) that for every t which verifies (2.18), there exists a subsequence of $\{u^{(n)}+iv^{(n)}\}_{n=0}^{\infty}$ which converges uniformly on the compact subsets of

$$\mathcal{D}_t=\left\{(x, y)\in\mathbb{C}^D: x\in B(x_0, d_0/4), |y|<\frac{1}{Cp^{1/2}}\min\{t^{1/2}, d_0\}\right\}$$

to the analytic extension of u. ∎

Now, assume

$$\|u_0\|_{L^\infty}\leqslant M_\infty$$

for some $M_\infty\geqslant 2$.

A function $u: \bar{\Omega}\times[0, T)\to\mathbb{R}$, where $T>0$, is a solution of the initial value problem (2.1)–(2.3) if $u\in L^\infty((0, T_0)\times\Omega)$, for every $T_0\in(0, T)$, if u satisfies (2.1) and (2.2) in the classical sense for $t\in(0, T)$, and if

$$\lim_{t\to 0^+}\|u(\cdot, t)-u_{\mathrm{lin}}(\cdot, t)\|_{L^\infty}=0$$

where u_{lin} is the solution of

$$\begin{aligned} \partial_t u-\Delta u&=0\quad, && t>0\\ u|_{\partial\Omega}&=0, && t>0\\ u(x, 0)&=u_0(x), && x\in\Omega. \end{aligned}$$

THEOREM 2.4. *Let* $x_0\in\Omega$ *and* $d_0=\mathrm{dist}(x_0, \partial\Omega)$. *For*

$$0<t\leqslant\frac{1}{C}\min\left\{d_0^2, \frac{1}{M_\infty{}^{k-1}\log M_\infty}\right\}$$

the analyticity radius $\delta_{x_0}(t)$ *of* u *at point* x_0 *at time* t *satisfies*

$$\delta_{x_0}(t)\geqslant\frac{1}{C(\log M_\infty)^{1/2}}\min\{t^{1/2}, d_0\}.$$

Proof. Note that

$$\|u_0\|_{L^{2p}}\leqslant CM_\infty, \qquad p\in[1/2, \infty).$$

The theorem follows from Theorem 2.1 by letting $p=(D(k-1)+2+\log M_\infty)/4$. ∎

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REFERENCES

[B] C. Bardos, Analyticité de la solution de l'équation d'Euler dans un ouvert de R^n, *C.R. Acad. Sci. Paris Sér. A-B* **283** (1976), A255–A258.

[BB] C. Bardos and S. Benachour, Domaine d'analyticité des solutions de l'équation d'Euler dans un ouvert de R^n, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **4** (1977), 647–687.

[BC] H. Brezis and T. Cazenave, A nonlinear heat equation with singular initial data, *J. Anal. Math.* **68** (1996), 277–304.

[CEES] P. Collet, J.-P. Eckmann, H. Epstein, and J. Stubbe, Analyticity for the Kuramoto–Sivashinsky equation, *Phys. D* **67** (1993), 321–326.

[CRT] C. Cao, M. A. Ramaha, and E. S. Titi, Gevrey regularity for nonlinear analytic parabolic equations on the sphere, preprint.

[DTH] L. Duan, E. S. Titi, and P. Holmes, Regularity, approximation and asymptotic dynamics for a generalized Ginzburg–Landau equation, *Nonlinearity* **6** (1993), 915–933.

[FT] A. B. Ferrari and E. S. Titi, Gevrey regularity for nonlinear analytic parabolic equations, *Comm. Partial Differential Equations* **23** (1998), 1–16.

[FoT] C. Foias and R. Temam, Gevrey class regularity for the solutions of the Navier–Stokes equations, *J. Funct. Anal.* **87** (1989), 359–369.

[G] Y. Giga, Time and spatial analyticity of solutions of the Navier–Stokes equations, *Comm. Partial Differential Equations* **8** (1983), 929–948.

[Gr] Z. Grujić, Space analyticity on the attractor generated by the set of all stationary solutions for the Kuramoto–Sivashinsky equation, *J. Dynam. Differential Equations*, to appear.

[GK] Z. Grujić and I. Kukavica, Space analyticity for the Navier–Stokes and related equations with initial data in L^p, *J. Funct. Anal.* **152** (1998), 447–466.

[HKR] W. D. Henshaw, H.-O. Kreiss, and L. G. Reyna, Estimates of the local minimum scale for the incompressible Navier–Stokes equations, *Numer. Funct. Anal. Optim.* **16** (1995), 315–344.

[J] F. John, "Partial Differential Equations," Fourth Edition, Springer-Verlag, New York, 1982.

[K] I. Kukavica, Level sets of the vorticity and the stream function for the 2D periodic Navier–Stokes equations with potential forces, *J. Differential Equations* **126** (1995), 374–388.

[KM] T. Kato and K. Masuda, Nonlinear evolution equations and analyticity, I, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **3** (1986), 455–467.

[LO] C. D. Levermore and M. Oliver, Analyticity of solutions for a generalized Euler equation, *J. Differential Equations* **133** (1997), 321–339.

[M1] K. Masuda, On the analyticity and the unique continuation theorem for solutions of the Navier–Stokes equation, *Proc. Japan Acad. Ser. A Math. Sci* **43** (1967), 827–832.

[M2] K. Masuda, On the regularity of solutions of the nonstationary Navier–Stokes equations, Lecture Notes in Mathematics, Vol. 771, 827–832, Springer-Verlag, New York, 1980.

[TBDVT] P. Takáč, P. Bollerman, A. Doelman, A. Van Harten, and E. S. Titi, Analyticity of essentially bounded solutions to semilinear parabolic systems and validity of the Ginzburg–Landau equation, *SIAM J. Math. Anal.* **27** (1996), 424–448.

[W] F. B. Weissler, Existence and nonexistence of global solutions for a semilinear heat equation, *Israel J. Math.* **38** (1981), 29–40.