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Spatial Analyticity on the Global Attractor for the Kuramoto–Sivashinsky Equation

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For the Kuramoto–Sivashinsky equation with L-periodic boundary conditions we show that the radius of space analyticity on the global attractor is lower-semicontinuous function at the stationary solutions, and thereby deduce the existence of a neighborhood in the global attractor of the set of all stationary solutions in which the radius of analyticity is *independent* of the bifurcation parameter L. As an application of the result, we prove that the number of rapid spatial oscillations of functions belonging to this neighborhood is, up to a logarithmic correction, at most *linear* in L.

KEY WORDS: Kuramoto–Sivashinsky equation; attractors; analyticity in the space variable; spatial chaos.

1. INTRODUCTION

We consider the Kuramoto–Sivashinsky equation (KSE) in one space dimension with L-periodic boundary conditions. L serves as the bifurcation parameter for the equation.

It is known that KSE generates a well-defined dynamical system in the appropriate phase space and that the functions belonging to the global attractor $\mathscr{A}$ are analytic in the space variable with the radius of analyticity proportional to $L^{-16/25}$ ([1, 2]). However, the numerical results presented in [2] strongly suggest that the radius of analyticity for large time, or equivalently on the global attractor, is essentially independent of L. In [6] J.-P. Eckmann, I. Kukavica and J. Stubbe observed that the radius of analyticity of a stationary solution is a universal quantity, depending neither on a particular stationary solution nor on L.

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Starting from their result, we obtain a neighborhood, in the global attractor, of the set of all stationary solutions in which the radius of analyticity is *independent* of the bifurcation parameter L. The proof shows that the analyticity radius on the global attractor is lower-semicontinuous function at the stationary solutions, a property that holds for other dissipative PDE's as well, e.g. 2D Navier-Stokes Equations, Ginzburg–Landau Equations... .

Also, in our use of the Gevrey-class technique [4] we introduce a new, equivalent norm—see Lemma 4.1 below—which may be helpful in the study of other dissipative PDE's, particularly the Navier-Stokes equations on closed manifolds.

Finally, utilizing the derived analyticity properties via the Jensen's formula, we obtain an upper bound on the number of rapid oscillations in the space variable of the functions belonging to the neighborhood. The bound is, up to a logarithmic factor, *linear* in L and thus, in the neighborhood, we have *linear* control on this particular manifestation of spatial chaos. For a different method of using the analyticity to estimate the number of spatial oscillations see [7].

The paper is organized as follows. Section 3 contains the results concerning the stationary solutions, while in Section 4 we introduce the analytic representation of the Gevrey norm. In Section 5 we obtain Gevrey estimates around a stationary solution and in Section 6 we recall some properties of the flow restricted to the global attractor. Section 7 contains the main results and in the last section we establish an upper bound on the number of rapid spatial oscillations.

Throughout this work *the attractor* will stand for the global attractor.

2. NOTATION

We consider the following problem for a real valued function $u = u(x, t)$, $x \in \mathbb{R}$, $t > 0$:

$$u_t + u_{xxxx} + u_{xx} + uu_x = 0 \tag{2.1}$$

$$u \text{ is } L\text{-periodic in } x \tag{2.2}$$

$$\int_0^L u(x, t)\, dx = 0 \tag{2.3}$$

First, we introduce the functional setting. The phase space of the equation is the Hilbert space

$$H = \left\{ u = \sum_{j=-\infty}^{\infty} u_j e^{iqjx} : |u|^2 = \int_0^L |u(x)|^2 \, dx \right.$$

$$\left. = L \sum_{j=-\infty}^{\infty} |u_j|^2 < \infty, \overline{u_j} = -u_j, u_0 = 0 \right\}$$

$q = 2\pi/L$, where the scalar product is given by

$$(u, v) = \int_0^L u(x)\, v(x)\, dx = L \sum_{j=-\infty}^{\infty} u_j \overline{v_j}$$

For $s > 0$, we consider the positive powers of $A = \partial^4/\partial x^4$

$$A^s u = \sum_{j=-\infty}^{\infty} (q\,|j|)^{4s}\, u_j e^{iqjx}$$

for

$$u \in D(A^s) = \left\{ u \in H : \sum_{j=-\infty}^{\infty} (q\,|j|)^{8s}\, |u_j|^2 < \infty \right\}$$

Also, for τ, $s > 0$, we consider the weighted exponential operators [4]

$$e^{\tau A^s} u = \sum_{j=-\infty}^{\infty} e^{\tau (q\,|j|)^{4s}}\, u_j e^{iqjx}$$

for

$$u \in D(e^{\tau A^s}) = \left\{ u \in H : \sum_{j=-\infty}^{\infty} e^{2\tau (q\,|j|)^{4s}}\, |u_j|^2 < \infty \right\}$$

$D(e^{\tau A^s})$ is a Hilbert space, the scalar product being simply

$$(u, v)_\tau = (e^{\tau A^s} u, e^{\tau A^s} v) \tag{2.4}$$

and it is well-known that for $s = 1/4$ u is the restriction of a function analytic in a strip of width τ in the complex plane containing the real axis. This type of analyticity will be referred to as τ-analyticity.

Now, (2.1)–(2.3) are equivalent to

$$u_t + Au - A^{1/2} u + B(u) = 0 \tag{2.5}$$

where $B(u) = B(u, u)$, and $B(u, v) = uv_x$.

It is well-known that for any $u_0 \in H$ there exists a unique continuous function $u(t)$ from $[0, \infty)$ to H, satisfying (2.5) for all $t > 0$. Therefore, one can define the solution semigroup of (2.5) by $S(t)\, u_0 = u(t)$ for all $t \geqslant 0$, $u_0 \in H$.

For convenience, throughout the exposition, $c(p_1, p_2, ..., p_n)$, or $c_{p_1, p_2, ..., p_n}$, will denote a constant depending on the parameters $p_1, p_2, ..., p_n$ which may differ from line to line.

3. ANALYTICITY FOR THE STATIONARY SOLUTIONS

The stationary KSE can be written as

$$u^*_{xxx} + u^*_x = -\alpha - 1/2(u^*)^2$$

where α is an integration constant. First, we note that $\alpha \leqslant 0$, integrating the equation from 0 to L, and so we can write

$$u^*_{xxx} + u^*_x = c^2 - 1/2(u^*)^2 \tag{3.1}$$

with $c \geqslant 0$. It is proved in [8], Theorem 2.2, that there exists a universal constant c^* such that (3.1) considered on the whole space has no periodic solutions for $c \geqslant c^*$. Besides, Lemma 2.1 [8] states that for $c \in [0, c_0]$, $c_0 > 0$ arbitrary, the set of all bounded solutions of (3.1) on $\mathbb{R}$ is together with their first two derivatives uniformly bounded by the constant depending only on c_0. Thus, for the periodic boundary conditions, we derive the following consequences [6].

There exists a universal constant K such that all stationary solutions u^* of (2.5) satisfy

$$\|u^*\|_\infty, \|u^*_x\|_\infty, \|u^*_{xx}\|_\infty \leqslant K$$

where we denoted by $\|u\|_\infty$ the L^∞-norm of u on $\mathbb{R}$, $\|u\|_\infty = \operatorname{ess\,sup}_{x \in \mathbb{R}} |u(x)|$. Starting from the above result one sets up a recursive relation, differentiating (3.1) repeatedly and setting $M = \max\{1, K, c^*\}$, leading to

$$\|u^*\|_\infty \leqslant M, \qquad \left\| \frac{d^n u^*}{dx^n} \right\|_\infty \leqslant M^n n!, \qquad n \geqslant 1 \tag{3.2}$$

Consequently, u^* has an analytic extension $u^*(z)$ such that

$$|u^*(z)| \leqslant M - 1 + \frac{1}{1 - M\,|z - x|}, \qquad |z - x| < 1/M \tag{3.3}$$

and thus

$$|u^*(z)|,\ |(u^*)'(z)| \leqslant c, \qquad |\Im z| \leqslant 1/(2M) \tag{3.4}$$

Note that M is by construction an absolute constant.

4. THE ANALYTIC REPRESENTATION OF THE GEVREY NORM

We introduce a norm, equivalent to the standard Gevrey norm $|u|_\tau = (u, u)_\tau^{1/2}$, see (2.4), which will simplify the estimates for the nonlinear terms involving u^* in the equation obtained from (2.5) for $u - u^*$.

For $s \in \mathbb{R}$ we consider a family of line segments

$$\Gamma_s = \{x + is,\ x \in [0, L]\}$$

Then, for $s \geqslant 0$, we define

$$((u, v))_s = \int_{\Gamma_s \cup \Gamma_{-s}} u\bar{v}\, dx = 2 \int_{\Gamma_s} \Re(u\bar{v})\, dx \tag{4.1}$$

for $u, v \in D(e^{sA^{1/4}})$.

Lemma 4.1. *(4.1) is well-defined. Moreover,* $\|u\|_s^2 = |u|_s^2 + |u|_{-s}^2$ *and thus* $\|u\|_s$ *is equivalent to the standard Gevrey norm* $|u|_s$.

Proof. Let u be in $D(e^{sA^{1/4}})$, for some $s > 0$. Then the complexified Galerkin expansion $u(z) = \sum_{j=-\infty}^{\infty} u_j e^{iqjz}$ provides an analytic extension of u to $\{z: |\Im z| < s\}$. To see that, it is enough to show that $u_N(z) = \sum_{j=-N}^{N} u_j e^{iqjz}$ converge uniformly on compact sets in $\{z: |\Im z| < s\}$. First, note that

$$\|u_N\|_{L^2(\Gamma_y)}^2 = \sum_{j=-N}^{N} |u_j|^2\, e^{-2qjy} \leqslant |u|_s^2$$

for $|y| \leqslant s$. Also,

$$\|u_N'\|_{L^2(\Gamma_y)}^2 = \sum_{j=-N}^{N} (qj)^2\, |u_j|^2\, e^{-2qjy} \leqslant e^{-2}(s - y_0)^{-2}\, |u|_s^2$$

for $|y| \leqslant y_0 < s$.

Now, we can apply the Agmon's inequality on every Γ_y, $|y| \leqslant y_0 < s$, and obtain that $\{u_N\}$ is uniformly bounded on $\{z: |\Im z| \leqslant y_0\}$, for all $y_0 < s$. By Montel's theorem, $\{u_N\}$ is a normal family on $\{z: |\Im z| < s\}$, and since we already have the uniform convergence on $\mathbb{R}$, the whole sequence converges uniformly on the compact sets.

Moreover, since $\sum_{j=-\infty}^{\infty} u_j e^{iqjz}$ converges uniformly in $L^2(\Gamma_y)$, for $|y| < s$, the following computation is valid—it is enough to consider only the case $0 < y < s$.

$$\begin{aligned}
\|u\|_y^2 &= 2\int_{\Gamma_y} |u|^2 \, dx \\
&= 2\int_0^L \left(\sum_{j=-\infty}^{\infty} u_j e^{iqj(x+iy)}\right)\overline{\left(\sum_{j=-\infty}^{\infty} u_j e^{iqj(x+iy)}\right)} dx \\
&= 2\sum_{j=-\infty}^{\infty} |u_j|^2 e^{-2qjy} \\
&= \sum_{j=-\infty}^{\infty} |u_j|^2 e^{-2qjy} + \sum_{j=-\infty}^{\infty} |u_j|^2 e^{2qjy} \qquad (u_{-j} = \overline{u_j}) \\
&= \sum_{j>0} |u_j|^2 e^{-2q|j|y} + \sum_{j<0} |u_j|^2 e^{2q|j|y} \\
&\quad + \sum_{j>0} |u_j|^2 e^{2q|j|y} + \sum_{j<0} |u_j|^2 e^{-2q|j|y} \\
&= \sum_{j=-\infty}^{\infty} |u_j|^2 e^{-2q|j|y} + \sum_{j=-\infty}^{\infty} |u_j|^2 e^{2q|j|y} \\
&= |u|_{-y}^2 + |u|_y^2
\end{aligned} \tag{4.2}$$

Next, we show that $u(\cdot + iy)$ has $L^2([0, L])$-limit as y converges to s.

Define $U(x) = \sum_{j=-\infty}^{\infty} u_j e^{iqj(x+is)}$, for $x \in [0, L]$. Then, $U \in L^2([0, L])$, since $\sum_{j=-\infty}^{\infty} |u_j|^2 e^{2q|j|s} = |u|_s^2 < \infty$, and

$$\|u(\cdot + iy) - U\|_{L^2([0,L])}^2 = \sum_{j=-\infty}^{\infty} |u_j|^2 (e^{-qjy} - e^{-qjs})^2$$

Since $|u|_s^2 < \infty$, the above series converges uniformly in y, and we can take the limit obtaining that $u(\cdot + iy)$ converges to U in $L^2([0, L])$; consequently, we can extend u to Γ_s as $L^2(\Gamma_s)$-function.

Finally, we can take the limit in (4.2) establishing the desired result. □

In the following lemma we obtain the estimates for the nonlinear term using the analytic representation of the Gevrey norm.

Lemma 4.2. *Let $u^* \in H$ have an analytic extension to $\{z: |\Im z| < \tau^*\}$ for some $\tau^* > 0$. Assume also that for some $\varepsilon \in (0, \tau^*)$, $w_{xx} \in D(e^{(\tau^* - \varepsilon) A^{1/4}})$ and let $0 < s \leqslant \tau^* - \varepsilon$.*

Then

$$|((B(w), w))_s| \leqslant 1/8 \ \|w''\|_s^2 + c \ \|w\|_s^{18/5} \tag{4.3}$$

$$|((B(u^*, w), w))_s| \leqslant 1/8 \ \|w''\|_s^2 + c_\varepsilon \ \|w\|_s^2 \tag{4.4}$$

$$|((B(w, u^*), w))_s| \leqslant c_\varepsilon \ \|w\|_s^2 \tag{4.5}$$

where for fixed u^*, c_ε *depends on* ε *through* $\max_{\{z:\, |\Im z| \leqslant \tau^* - \varepsilon\}} \{|u^*(z)|, |(u^*)'(z)|\}$.

Proof. For (4.3),

$$\begin{aligned}
|((B(w), w))_s| &= \left| \int_{\Gamma_s \cup \Gamma_{-s}} |w|^2 \, w' \, dx \right| \\
&\leqslant \operatorname*{ess\ sup}_{\Gamma_s \cup \Gamma_{-s}} |w| \left(\int_{\Gamma_s \cup \Gamma_{-s}} |w|^2 \, dx \right)^{1/2} \left(\int_{\Gamma_s \cup \Gamma_{-s}} |w'|^2 \, dx \right)^{1/2} \\
&\leqslant \left(\int_{\Gamma_s \cup \Gamma_{-s}} |w|^2 \, dx \right)^{3/4} \left(\int_{\Gamma_s \cup \Gamma_{-s}} |w'|^2 \, dx \right)^{3/4} \\
&\leqslant c \left(\int_{\Gamma_s \cup \Gamma_{-s}} |w|^2 \, dx \right)^{9/8} \left(\int_{\Gamma_s \cup \Gamma_{-s}} |w''|^2 \, dx \right)^{3/8} \\
&\leqslant 1/8 \ \|w''\|_s^2 + c \ \|w\|_s^{18/5}
\end{aligned}$$

where we used Agmon's, interpolation and Young's inequalities. For (4.4),

$$\begin{aligned}
|((B(u^*, w), w))_s| &= \left| \int_{\Gamma_s \cup \Gamma_{-s}} u^* w' \bar{w} \, dx \right| \\
&\leqslant \max_{\Gamma_s \cup \Gamma_{-s}} |u^*| \left(\int_{\Gamma_s \cup \Gamma_{-s}} |w'|^2 \, dx \right)^{1/2} \\
&\leqslant c_\varepsilon \left(\int_{\Gamma_s \cup \Gamma_{-s}} |w|^2 \, dx \right)^{3/4} \left(\int_{\Gamma_s \cup \Gamma_{-s}} |w''|^2 \, dx \right)^{1/4} \\
&\leqslant 1/8 \ \|w''\|_s^2 + c_\varepsilon \ \|w\|_s^2
\end{aligned}$$

Finally, for (4.5),

$$\begin{aligned}
|((B(w, u^*), w))_s| &= \left| \int_{\Gamma_s \cup \Gamma_{-s}} (u^*)' \, |w|^2 \, dx \right| \\
&\leqslant \max_{\Gamma_s \cup \Gamma_{-s}} |(u^*)'| \left(\int_{\Gamma_s \cup \Gamma_{-s}} |w|^2 \, dx \right) \leqslant c_\varepsilon \ \|w\|_s^2
\end{aligned}$$

□

5. THE STATIONARY SOLUTIONS GENERATE ANALYTICITY

For a stationary solution u^* of (2.5), let τ^* denote the maximum $\tau > 0$ such that u^* is τ-analytic; note that $\tau^* \geqslant 1/M$. The following is the main result of this section.

Theorem 5.1. *For every $\varepsilon \in (0, \tau^*)$, there exist ρ^*_ε, T^*_ε, $K^*_\varepsilon > 0$ such that*

$$|e^{(\tau^* - \varepsilon) A^{1/4}}(S(T^*_\varepsilon)\, u_0 - u^*)| \leqslant K^*_\varepsilon\, |u_0 - u^*|$$

for any $u_0 \in H$ satisfying $|u_0 - u^*| \leqslant \rho^*_\varepsilon$.

Remark 5.1. It will be shown in the proof that ρ^*_ε, T^*_ε, and K^*_ε depend on a particular u^* and ε through $\max_{\{z:\, |\Im z| \leqslant \tau^* - \varepsilon\}} \{|u^*(z)|, |(u^*)'(z)|\}$.

Proof. Let $w = u - u^*$. Then w satisfies the following equation

$$w_t + w_{xxxx} + w_{xx} + w w_x + u^* w_x + u^*_x w = 0 \tag{5.1}$$

We take the scalar product $((\ ,\ ))_s$, introduced in (4.1), of the equation (5.1) with w for $s = \alpha t \leqslant \tau^* - \varepsilon$, α being a fixed positive number. For the linear terms, we use the Gevrey representation of $((\ ,\ ))_s$, and for the bilinear terms we apply the direct estimates from Lemma 4.2. That leads to

$$\begin{aligned}\frac{d}{dt}\, \|w\|^2_{\alpha t} + 2\, \|A^{1/2} w\|^2_{\alpha t} \leqslant{}& \|A^{1/4} w\|^2_{\alpha t} + 2\alpha\, \|A^{1/8} w\|^2_{\alpha t} + 2\, |((B(w), w))_{\alpha t}| \\ &+ 2\, |((B(u^*, w), w))_{\alpha t}| + 2\, |((B(w, u^*), w))_{\alpha t}|\end{aligned} \tag{5.2}$$

and thus

$$\frac{d}{dt}\, \|w\|^2_{\alpha t} \leqslant \{c(1 + \alpha^{4/3} + c_\varepsilon) + c\, \|w\|^{8/5}_{\alpha t}\}\, \|w\|^2_{\alpha t} \tag{5.3}$$

We multiply (5.3) by $e^{-\eta_\varepsilon t}$, for $\eta_\varepsilon = 1 + c(1 + \alpha^{4/3} + c_\varepsilon)$, and set $x(t) = e^{-\eta_\varepsilon t}\, \|w\|^2_{\alpha t}$.

Then, we can rewrite (5.3) as

$$\dot{x} + (1 - c e^{(4/5)\, \eta_\varepsilon t} x^{4/5})\, x \leqslant 0 \tag{5.4}$$

Let $T_\varepsilon = (\tau^* - \varepsilon)/\alpha$. Then

$$\dot{x} + (1 - c e^{(4/5)\, \eta_\varepsilon T_\varepsilon} x^{4/5})\, x \leqslant 0, \qquad t \in (0, T_\varepsilon] \tag{5.5}$$

Thus, if $|u_0 - u^*| \leqslant \rho_\varepsilon^*$, for suitable ρ_ε^*, we obtain the statement of the theorem. Since $\alpha > 0$ is still a free parameter we choose it to maximize

$$\rho_\varepsilon^* = c \frac{1}{e^{[1 + c(1 + \alpha^{4/3} + c_\varepsilon)][(\tau^* - \varepsilon)/\alpha]}}$$

□

Remark 5.2. The estimates in the proof are formal, but they can be made rigorous considering the Galerkin approximations.

6. SOME REMARKS ON THE FLOW RESTRICTED TO THE ATTRACTOR

It is well known that the semigroup S restricted to the attractor is actually a group and $S(t)^{-1} = S(-t)$, $t \in \mathbb{R}$. Moreover, $S(\cdot)\colon \mathscr{A} \to \mathscr{A}$ is a homeomorphism [11, 3]. Also, from the general theory [11], it follows that $\mathscr{A}$ is connected.

Now, denote by S^* the set of all fixed points of S.

Proposition 6.1. *Let ρ, $T > 0$, $u^* \in S^*$. Then,*

$$\mathscr{N}_{u^*}(\rho, T) = S(T)(B(u^*, \rho) \cap \mathscr{A})$$

is an open and connected neighborhood of u^* in $\mathscr{A}$.

The proposition follows directly from the above remarks.

7. THE MAIN RESULTS

We are now ready to state and prove our main results.

Theorem 7.1 (the lower-semicontinuity of the radius of analyticity on the attractor at the stationary solutions). *Let u^* be a stationary solution of (2.5) and let τ^* denote the maximum $\tau > 0$ such that u^* is τ-analytic; recall that $\tau^* \geqslant 1/M$. Then, for every $\varepsilon \in (0, \tau^*)$, there exists $\mathscr{N}_{u^*}(\varepsilon)$ such that every u belonging to $\mathscr{N}_{u^*}(\varepsilon)$ is $(\tau^* - \varepsilon)$-analytic and $\mathscr{N}_{u^*}(\varepsilon)$ is an open and connected neighborhood of u^* in $\mathscr{A}$.*

Proof. This is a direct consequence of Theorem 5.1 and Proposition 6.1. Define $\mathscr{N}_{u^*}(\varepsilon)$ to be $\mathscr{N}_{u^*}(\rho_\varepsilon^*, T_\varepsilon^*)$, where $\mathscr{N}_{u^*}(\cdot, \cdot)$ is as in Proposition 6.1 and ρ_ε^*, T_ε^* as in Theorem 5.1. □

Theorem 7.2. *In the attractor, there exists an open neighborhood $\mathcal{N}$ of the set of all stationary solutions S^* in which the radius of analyticity is independent of L.*

Proof. In Lemma 4.2, for any stationary solution u^*, set $\tau^* = 1/M$, $\varepsilon = 1/(2M)$ and apply the uniform estimates (3.4). Then, the constants ρ_ε^* and T_ε^* in Theorem 5.1 become universal and we can define $\mathcal{N} = \bigcup_{u^* \in S^*} \mathcal{N}_{u^*}(1/(2M))$, where $\mathcal{N}_{u^*}(\cdot)$ is as in Theorem 7.1. $\square$

8. AN APPLICATION OF THE JENSEN'S FORMULA

We can assume that $L \geqslant 2\pi$, otherwise $\mathcal{A} = \{0\}$. The purpose of this section is to prove the following result.

Theorem 8.1. *Let $\mathcal{N}$ be the neighborhood in $\mathcal{A}$ of the set of all stationary solutions introduced in Theorem 7.2. Then, for $u \in \mathcal{N}$, $[0, L] = I \cup R$, where I is a union of at most $[cL]$ intervals open in $[0, L]$, and*

(i) $|u'(x)| < 1/L$, *for all* $x \in I$,

(ii) $\operatorname{card}\{x \in R : u'(x) = 0\} \leqslant cL \log L$.

For the proof we need two lemmas. The first lemma is an easy consequence of the Jensen's formula (see [9, 15.18 and 15.20]).

Lemma 8.1. *Let $z_0 \in \mathbb{C}$, $R > 0$, and let u be analytic in the neighborhood of $\{z\colon |z - z_0| \leqslant 2R\}$ and $u'(z_0) \neq 0$. Then*

$$\operatorname{card}\{z\colon |z - z_0| \leqslant R,\, u'(z) = 0\} \leqslant \frac{1}{\log 2} \log \frac{\max_{|z - z_0| = 2R} |u'(z)|}{|u'(z_0)|}$$

Lemma 8.2. *Let L, $\tau > 0$, and let u be analytic in the neighborhood of $\{z\colon |\Im z| \leqslant \tau\}$ and L-periodic in x-direction. Then, for any $\varepsilon > 0$, $[0, L] = I_\varepsilon \cup R_\varepsilon$, where I_ε is a union of at most $[2\frac{L}{\tau}]$ intervals open in $[0, L]$, and*

(i) $|u'(x)| < \varepsilon$, *for all* $x \in I_\varepsilon$,

(ii) $\operatorname{card}\{x \in R_\varepsilon : u'(x) = 0\} \leqslant (2/\log 2)\ L/\tau \log(\max_{|\Im z| \leqslant \tau} |u'(z)|/\varepsilon)$.

Proof. Let $x_1 = \inf\{x \in [0, L] : |u'(x)| \geqslant \varepsilon\}$. Then, by Lemma 8.1

$$\operatorname{card}\{x \in [x_1 - \tau/2, x_1 + \tau/2] : u'(x) = 0\} \leqslant \frac{1}{\log 2} \log \frac{\max_{|\Im z| \leqslant \tau} |u'(z)|}{\varepsilon}$$

Next, we define $x_2 = \inf\{x \in [x_1 + \tau/2, L] : |u'(x)| \geqslant \varepsilon\}$ and by applying Lemma 8.1 again we get exactly the same estimate on $[x_2 - \tau/2, x_2 + \tau/2]$. We can continue this process at most $N = 2L/\tau$ times and finally obtain the lemma for $R_\varepsilon = (\bigcup_{i=1}^{N} [x_i - \tau/2, x_i + \tau/2]) \cap [0, L]$ and $I_\varepsilon = [0, L] - R_\varepsilon$. □

The proof of Theorem 8.1. By the construction of the neighborhood $\mathscr{N}$, every function u belonging to $\mathscr{N}$ is analytic at least in the strip of width 2τ, where τ is an absolute constant and $|u(z)| \leqslant c$ in $\{z \colon |\Im z| < 2\tau\}$. Hence, by the Cauchy formula, u' is uniformly bounded by an absolute constant on the closed strip of width τ. Now, we can apply Lemma 8.2 with $\varepsilon = 1/L$ and obtain the statement of the theorem. □

Remark 8.1. The upper bounds on the cardinality of the zero sets of solution, and their derivatives were established in [7]. In particular, it was shown that for any $u \in \mathscr{A}$, the number of spatial oscillations is universally bounded by an expression that is essentially exponential in L. If we apply our method to arbitrary $u \in \mathscr{A}$, we get that the number of rapid spatial oscillations is universally bounded by an expression that is algebraic in L, namely $L^{41/25} \log L$. In the other words, if we ignore the essentially flat portions of the graph of u, the number of oscillations is at most *algebraic*.

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