A Family of Regularity Classes for the 3D NSE Approximating a Critical Class

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ABSTRACT. Exploiting the localization of the vortex-stretching mechanism to small spatio-temporal scales obtained in [10] and geometric structure of the leading-order vortex-stretching term, a family of regularity classes approximating a critical, NSE-scaling invariant class is obtained.

1. INTRODUCTION

Let u be a weak solution to the 3D NSE

$$u_t - \Delta u + (u \cdot \nabla)u + \nabla p = 0, \tag{1.1}$$

supplemented with the incompressibility condition $\operatorname{div} u = 0$, on an open subset of the space time $D = \Omega \times (0,T)$. A function class $\mathcal{F}$ on D is a regularity class for the 3D NSE if boundedness in $\mathcal{F}$ implies local boundedness (in L^∞) of u on $\Omega \times (0,T]$.

A classical regularity class is the $L^q_x L^p_t$ Foias-Prodi-Serrin class requiring

$$||u||_q^{2q/(q-3)} \in L^1(0,T) \quad \text{for some } 3 < q \le \infty;$$

for a proof of local boundedness as well as generalizations to the endpoint case $q = 3$ and to the weak Lebesgue spaces see [13, 16–18] and the references therein.

An analogous regularity class pertaining Du was obtained in [7],

$$||Du||_q^{2q/(2q-3)} \in L^1(0,T) \quad \text{for some } 3 \le q < \infty;$$

the case $q = \infty$ follows from a different argument. In fact, the time integrability of $\|\omega\|_\infty$, where $\omega = \operatorname{curl} u$ is the vorticity, is a well-known Beale-Kato-Majda criterion [1] (actually, less is required—the time integrability of the BMO-norm of ω [14]).

All of the aforementioned regularity classes are invariant with respect to the NSE scaling. More precisely, if $u(x,t)$ is a solution to the 3D NSE on, say $\mathbb{R}^3 \times (0,\infty)$, then $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$, $\lambda > 0$, is a solution as well, and a regularity class-norm of the rescaled solution u_λ coincides with the norm of the original solution u. A local version of this property is realized by rescaling parabolic space-time cylinders $Q_r(x_0,t_0) = B_r(x_0) \times (t_0 - r^2, t_0)$, $r > 0$, around a space-time point (x_0,t_0) in $\Omega \times (0,T)$.

A *geometric* approach to studying regularity in the 3D incompressible flows was pioneered by Constantin [4] where a singular integral representation of the stretching factor in the evolution of the vorticity magnitude was derived. The representation formula involves an explicit geometric kernel which is depleted by *local coherence* of the vorticity direction field.

This depletion mechanism was exploited in [5] to show that as long as the vorticity direction (in the region of intense vorticity) is Lipschitz-coherent, the solution remains regular. Following [5], it was shown in [8] that $\frac{1}{2}$-Hölder coherence suffices, and in [11], a class of hybrid geometric-analytic conditions for regularity was obtained containing a purely geometric $\frac{1}{2}$-Hölder coherence and a purely analytic Beale-Kato-Majda condition as the endpoint cases.

Restricting a representation formula for the vortex-stretching term on small, medium and large *spatial* scales, it was shown in [15] that the contribution of small scales is bounded by the dissipation, and that the contribution of the large scales is controlled by the known *a priori* estimates on the weak solutions. This result implied that, as long as the intense vorticity accumulates on small, sparsely populated sets, the solution remains regular. It is interesting that the *dissipation* scale transpired in this work is the same dissipation scale that appeared in the previous work [9] based on a totally different method (utilizing the sparseness of the vorticity super-level sets via a plurisubharmonic measure maximum principle in $\mathbb{C}^3$), and can be viewed as a localized vorticity-version of the Kolmogorov dissipation scale.

In a very recent work [10], a *localized* representation formula for *the vortex-stretching* term was obtained yielding localization of the evolution of the enstrophy to an arbitrarily small space-time cylinder $Q_r(x_0,t_0) = B_r(x_0) \times (t_0 - r^2, t_0)$. The proof merged the localization of the transport of the vorticity by the velocity previously obtained in [12] with the newly obtained localization of vortex-stretching (the localization of vortex-stretching presented in [12] utilized the singular integral representation formula for the vortex-stretching factor *over the whole space* $\mathbb{R}^3$ which was then split into small and large scales). The localization then lead to the regularity of *any* weak solution with a $\frac{1}{2}$-Hölder coherent vorticity direction field *independently* of the type of the spatial domain or the boundary conditions.

In this paper, the aforementioned localization of the vortex-stretching mechanism coupled with a geometric structure of the leading-order vortex-stretching term is utilized to show that boundedness of the gradient of the vorticity of a weak solution in a family of $L^q_x L^p_t$-classes approximating a critical, scaling invariant class suffices to control the localized enstrophy preventing the singularity formation. For any μ, $0 < \mu < 1$, sufficiently close to 1, there exists $\varepsilon^*(\mu) > 0$ such that for each ε, $0 < \varepsilon \le \varepsilon^*(\mu)$, the class in view is given by

$$||\nabla \omega||^{2q/((3-\varepsilon)q-3)}_{L^q(B(x_0,r))} \in L^1(t_0 - r^2,\, t_0) \quad \text{for some } q,\ \frac{3}{3-\varepsilon} < q < \frac{3\mu}{2-\mu\varepsilon}.$$

Notice that the limit class

$$||\nabla \omega||^{2q/(3q-3)}_{L^q(B(x_0,r))} \in L^1(t_0 - r^2,\, t_0) \quad \text{for some } q,\ 1 < q < \frac{3}{2},$$

is scaling invariant.

Another way of obtaining local $L^q_x L^p_t$-regularity conditions in terms of the vorticity would be utilizing local regularity conditions in terms of the velocity, e.g., [19], via local regularity of the system $\Delta u = -\operatorname{curl} \omega$, i.e., via the localized Bio-Savart law (cf. [10, Section 3]) and interpolating the low order terms between the *a priori* bounds on the weak solutions and the regularity condition. In contrast, the approach presented here is self-contained and illustrates a possibility of working with the localized vorticity without any reference to the velocity.

2. Localized Evolution of the Enstrophy

The vorticity formulation of the Navier-Stokes equations on a space-time domain $\Omega \times (0, T)$ reads

$$\omega_t - \Delta\omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u. \tag{2.1}$$

The right-hand side is the vortex-stretching term which holds a key to controlling the evolution of the enstrophy, $\|\omega(t)\|_2^2$, and consequently a key to preventing the singularity formation in the model.

Fix a point (x_0, t_0) in $\Omega \times (0, T)$ and let $\psi(x, t) = \varphi(x)\eta(t)$ be a smooth cut-off function on $Q_{2r}(x_0, t_0)$ satisfying

$$\operatorname{supp}\varphi \subset B(x_0, 2r), \qquad \varphi = 1 \text{ on } B(x_0, r),$$

$$\frac{|\nabla\varphi|}{\varphi^\rho} \le \frac{c}{r} \quad \text{for some } \rho \in (0,1), \qquad 0 \le \varphi \le 1,$$

and

$$\operatorname{supp}\eta \subset (t_0 - (2r)^2, t_0], \qquad \eta = 1 \text{ on } [t_0 - r^2,\, t_0],$$

$$|\eta'| \le \frac{c}{r^2}, \qquad 0 \le \eta \le 1.$$

It was shown in [12] and [10] that, choosing ρ sufficiently close to 1, it is possible to control *the lower order terms* in the localized transport term

$$\int_{Q_{2r}} (u \cdot \nabla)\omega \cdot \psi^2 \omega \, dx \, dt$$

(after integration by parts, the leading order transport term vanishes due to the incompressibility of the fluid) and in the localized vortex-stretching term

$$\int_{Q_{2r}} (\omega \cdot \nabla)u \cdot \psi^2 \omega \, dx \, dt,$$

respectively.

The following localization formula for the vortex-stretching term was obtained in [10],

$$\begin{aligned} (2.2) \qquad & \varphi^2(x)(\omega \cdot \nabla)u \cdot \omega(x) \\ &= \varphi(x)\frac{\partial}{\partial x_i} u_j(x)\varphi(x)\omega_i(x)\omega_j(x) \\ &= -cP.V. \int_{B(x_0,2r)} (\omega(x) \times \omega(y)) \cdot G_\omega(x,y)\varphi(y)\varphi(x) \, dy + \mathrm{VST}_{\mathrm{lot}} \\ &= \mathrm{VST} + \mathrm{VST}_{\mathrm{lot}} \end{aligned}$$

for x in $B(x_0, 2r)$, uniformly in time, where

$$(G_\omega(x,y))_k = \frac{\partial^2}{\partial x_i \partial y_k} \frac{1}{|x-y|} \omega_i(x)$$

and $\mathrm{VST}_{\mathrm{lot}}$ denotes the terms that are either lower order for at least one order of the differentiation or/and less singular for at least one power of $|x-y|$ than the leading order term VST.

Let u be a weak solution on $\Omega \times (0,T)$, and let $0 < R < 1$ be such that $Q_{2R}(x_0,t_0) \subset \Omega \times (0,T)$. As in [10, 12], for simplicity of the exposition, we assume that u is smooth in an open parabolic cylinder $Q_{2R}(x_0,t_0)$ and obtain bounds on the enstrophy localized to $B_R(x_0,t_0)$ *uniformly* in $(t_0 - R^2, t_0)$. This assumption can be avoided, for example, considering a class of suitable weak solutions for which the initial vorticity is a finite Radon measure constructed as limits of smooth time-delayed approximations by Constantin in [3].

Fix $r \le R$ and let s in $(t_0 - (2r)^2, t_0)$. The following estimate was derived in [12]:

$$\frac{1}{2}\int_{B(x_0,2r)} \varphi^2(x)|\omega|^2(x,s)\,\mathrm{d}x + \int_{Q_{2r}^s} |\nabla(\psi\omega)|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\le \int_{Q_{2r}} (|\eta|\,|\partial_t\eta| + |\nabla\psi|^2)|\omega|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \left|\int_{Q_{2r}^s} (u\cdot\nabla)\omega\cdot\psi^2\omega\,\mathrm{d}x\,\mathrm{d}t\right| + \left|\int_{Q_{2r}^s} (\omega\cdot\nabla)u\cdot\psi^2\omega\,\mathrm{d}x\,\mathrm{d}t\right|$$
$$= T_1 + T_2 + T_3$$
$$\le \frac{1}{2}\int_{Q_{2r}} |\nabla(\psi\omega)|^2\,\mathrm{d}x\,\mathrm{d}t + c(r)\int_{Q_{2r}} |\omega|^2\,\mathrm{d}x\,\mathrm{d}t + T_3$$

where $Q_{2r}^s = B(x_0,2r)\times(t_0-(2r)^2,s)$. (The localized transport term is first integrated by parts; the resulting leading order term vanishes due to the incompressibility and the lower order term is then estimated choosing the cut-off parameter ρ, $\frac{1}{2}\le\rho<1$).

Notice that

$$T_3 \le \left|\int_{Q_{2r}^s} \eta^2\,\mathrm{VST}\,\mathrm{d}x\,\mathrm{d}t\right| + \left|\int_{Q_{2r}^s} \eta^2\,\mathrm{VST}_{\mathrm{lot}}\,\mathrm{d}x\,\mathrm{d}t\right|.$$

Choosing ρ sufficiently close to 1, the second term (the sum of all the lower order vortex-stretching terms) can be bounded (cf. [10]) by

$$\max\left\{c\|\nabla u\|_{L^2(Q_{2r})}, \frac{1}{4}\right\}\left(\frac{1}{2}\sup_{t\in(t_0-(2r)^2,t_0)} ||\varphi\omega(t)||^2_{L^2(B(x_0,2r))} + ||\nabla(\psi\omega)||^2_{L^2(Q_{2r})}\right) + M(r,\|\nabla u\|_{L^2(Q_{2r})}).$$

It was shown in [10] that $\frac{1}{2}$-Hölder coherence of the vorticity direction field depletes the leading order vortex-stretching term preventing the singularity formation.

Here, we take a different approach: instead of assuming coherence of the vorticity direction, we will utilize the geometric structure of the leading order vortex-stretching term to rewrite it in a suitable form and then show that boundedness of $\nabla\omega$ in a family of $L^q_x L^p_t$-classes approximating a critical, scaling invariant class prevents the formation of singularities.

3. A Family of Regularity Classes Approximating a Critical Class

Theorem 3.1. *Let* u *be a weak solution on a space-time domain* $\Omega\times(0,T)$*. For any* μ, $0<\mu<1$, *sufficiently close to* 1, *there exists* $\varepsilon^*(\mu)>0$ *such that for each* ε, $0<\varepsilon\le\varepsilon^*(\mu)$, *the following is true.*

Fix a point (x_0, t_0) *in* $\Omega \times (0, T)$*, and let* $0 < R < \frac{1}{4}$ *be such that* $Q_{2R}(x_0, t_0) = B(x_0, 2R) \times (t_0 - (2R)^2, t_0)$ *is contained in* $\Omega \times (0, T)$*. Suppose that*

$$||\nabla \omega||_{L^q(B(x_0,2R))}^{2q/((3-\varepsilon)q-3)} \in L^1(t_0 - (2R)^2, t_0) \quad \text{for some } q,\ \frac{3}{3-\varepsilon} < q < \frac{3\mu}{2-\mu\varepsilon}.$$

Let u *be smooth in the open parabolic cylinder* $Q_{2R}(x_0, t_0)$*. Then the localized enstrophy remains* uniformly *bounded up to* $t = t_0$*, i.e.,*

$$\sup_{t\in(t_0-R^2,t_0)} \int_{B(x_0,R)} |\omega|^2(x,t)\,\mathrm{d}x < \infty.$$

Remark 3.2. As already mentioned, the assumption on u being smooth in the open cylinder can be avoided if u is, for example, a suitable weak solution constructed as a limit of a family of retarded mollifications (cf. [3]). Then the calculations presented in [12, pp. 558-559] can be carried out on the smooth approximations and the resulting bounds are inherited by the limit weak solutions.

Proof. Let $r \le R$. Recall that the leading order vortex-stretching term is given by

$$\text{(3.1)} \quad \int_{Q_{2r}^s} \eta^2(t)\,\text{VST}\,\mathrm{d}x\,\mathrm{d}t = c \int_{Q_{2r}^s} P.V. \int_{B(x_0,2r)} (\omega(x,t) \times \omega(y,t)) \times G_\omega(x,y,t)\psi(y,t)\psi(x,t)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t$$

where

$$(G_\omega(x,y,t))_k = \frac{\partial^2}{\partial x_i \partial y_k} \frac{1}{|x-y|}\omega_i(x,t).$$

Writing $\omega(y,t) = \omega(x,t) + (\omega(y,t) - \omega(x,t))$, the above expression can be rewritten as

$$c \int_{Q_{2r}^s} P.V. \int_{B(x_0,2r)} (\omega(x,t) \times (\omega(y,t) - \omega(x,t))) \times G_\omega(x,y,t)\psi(y,t)\psi(x,t)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t;$$

this yields the following bound

$$c \int_{Q_{2r}} P.V. \int_{B(x_0,2r)} \frac{1}{|x-y|^3} |\omega(y,t) - \omega(x,t)| \times \psi(y,t)\,\mathrm{d}y\,\psi(x,t)|\omega(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t = I.$$

Before continuing the estimate, we need to morph as much y-localization into x-localization as possible. Start with writing $\psi(y,t) = \psi(x,t) + (\psi(y,t) - \psi(x,t))$. This leads to

$$I \le c \int_{Q_{2r}} P.V. \int_{B(x_0,2r)} \frac{1}{|x-y|^3} |\omega(y,t) - \omega(x,t)| \, dy \, |(\psi\omega)(x,t)|^2 \, dx \, dt$$

$$+ c \int_{Q_{2r}} P.V. \int_{B(x_0,2r)} \frac{1}{|x-y|^3} |\omega(y,t) - \omega(x,t)|$$

$$\times |\psi(y,t) - \psi(x,t)| \, dy \, \psi(x,t) |\omega(x,t)|^2 \, dx \, dt$$

$$= I_1 + I_2.$$

Applying the Fundamental Theorem of Calculus to each ω_i,

$$\omega_i(y,t) - \omega_i(x,t) = \int_0^1 \nabla \omega_i((1-\lambda)x + \lambda y, t) \cdot (y - x) \, d\lambda,$$

and introducing the change of variables $z = x + \lambda(y - x)$, the following bound transpires

$$P.V. \int_{B(x_0,2r)} \frac{1}{|x-y|^3} |\omega(y,t) - \omega(x,t)| \, dy$$

$$\le c \frac{1}{\varepsilon} \int_{B(x_0,2r)} \frac{1}{|x-z|^{2+\varepsilon}} |\nabla \omega(z,t)| \, dz,$$

for any $0 < \varepsilon < 1$. Consequently,

$$I_1 \le c \frac{1}{\varepsilon} \int_{Q_{2r}} \int_{B(x_0,2r)} \frac{1}{|x-z|^{2+\varepsilon}} |\nabla \omega(z,t)| \, dz \, |(\psi\omega)(x,t)|^2 \, dx \, dt,$$

for any $0 < \varepsilon < 1$. For simplicity, we present the estimates on the integral in the case $\varepsilon = 0$; the result for a small ε will follow by changing the spatial integrability index variable q.

Let q be in $(1,3)$. Applying Hölder (in x) with the exponents

$$r = \frac{3q}{3-q} \quad \text{and } s = \frac{3q}{4q-3}$$

and then the weak Young with the exponents

$$1 + \frac{1}{r} = \frac{2}{3} + \frac{1}{q}$$

to the first factor yields

$$I_1 \le c \int_{t_0-(2r)^2}^{t_0} \|\nabla\omega(t)\|_{L^q(B(x_0,2r))} \, ||(\psi\omega)(t)||^2_{L^{2s}(\mathbb{R}^3)} \, \mathrm{d}t.$$

Interpolating the second factor in the integral,

$$I_1 \le c \int_{t_0-(2r)^2}^{t_0} \|\nabla\omega(t)\|_{L^q(B(x_0,2r))} \, ||(\psi\omega)(t)||^{(3q-3)/q}_{L^2(\mathbb{R}^3)} \, ||\nabla(\psi\omega)(t)||^{(3-q)/q}_{L^2(\mathbb{R}^3)} \, \mathrm{d}t.$$

Applying Hölder with the exponents

$$p_1 = \frac{2q}{3q-3}, \quad p_2 = \infty, \quad p_3 = \frac{2q}{3-q}$$

implies the final bound on I_1,

$$I_1 \le c\left(\int_{t_0-(2r)^2}^{t_0} ||\nabla\omega(t)||^{2q/(3q-3)}_{L^q(B(x_0,2r))} \right)^{(3q-3)/(2q)} \times \left(\frac{1}{2} \sup_{t\in(t_0-(2r)^2,t_0)} ||\varphi\omega(t)||^2_{L^2(B(x_0,2r))} + ||\nabla(\psi\omega)||^2_{L^2(Q_{2r})} \right).$$

Back to I_2. Similarly as before,

$$P.V. \int_{B(x_0,2r)} \frac{1}{|x-y|^3} |\omega(y,t)-\omega(x,t)| \, |\psi(y,t)-\psi(x,t)| \, \mathrm{d}y \le c\frac{1}{\varepsilon^2} \int_{B(x_0,2r)} \frac{1}{|x-z|^{1+2\varepsilon}} |\nabla\omega(z,t)| \, |\nabla\psi(z,t)| \, \mathrm{d}y$$

for any $0 < \varepsilon < 1$.

Let $\frac{1}{2} \le \rho < 1$. Then (cf. [10]),

$$|\nabla\psi(z,t)| \le c(\rho)\frac{1}{r}\psi^\rho(x,t) + c(r,\rho)|x-z|.$$

This yields

$$\begin{aligned} I_2 &\le c(\rho)\frac{1}{\varepsilon^2} \int_{Q_{2r}} \int_{B(x_0,2r)} \frac{1}{|x-z|^{1+2\varepsilon}} |\nabla\omega(z,t)| \, \mathrm{d}z \\ &\quad \times \left(\frac{1}{r}|\omega(x,t)|^{1-\rho} \right) |(\psi\omega)(x,t)|^{1+\rho} \, \mathrm{d}x \, \mathrm{d}t \\ &\quad + c(r,\rho)\frac{1}{\varepsilon^2} \int_{Q_{2r}} \int_{B(x_0,2r)} \frac{1}{|x-z|^{2\varepsilon}} |\nabla\omega(z,t)| \, \mathrm{d}z \\ &\quad \times |\omega(x,t)| \, |(\psi\omega)(x,t)| \, \mathrm{d}x \, \mathrm{d}t = A + B \end{aligned}$$

for any $0 < \varepsilon < 1$. As for I_1, we estimate the integrals in the case $\varepsilon = 0$ and address the case of arbitrary small ε at the end.

Note that the structure of A and B is completely analogous to the structure of the lower order terms A and B in [10]—the only difference being that ∇u is now replaced by $\nabla\omega$ and the the powers of $|x-z|$ are for one power less singular.

For A, notice that were the limit case $\rho = 1$ possible, we would have an expression of the form

$$\int_{Q_{2r}} \int_{B(x_0,2r)} \frac{1}{|x-z|} |\nabla\omega(z,t)| \,dz \,|(\psi\omega)(x,t)|^2 \,dx\,dt;$$

the same integrand as in I_1 except for one power of $|x-z|$ less singular. Consequently, we can choose ρ (sufficiently close to 1) in such a way that separating out the $((1/r)|\omega(x,t)|^{1-\rho})$ factor (and estimating it by a quantity depending only on r and $\|\omega\|_{L^2(Q_{2r})}$) leads to a desired bound.

For B, simply write

$$\begin{aligned}
B &\le c_1(r,\rho) \int_{t_0-(2r)^2}^{t_0} \|\nabla\omega(t)\|_{L^1(B(x_0,2r))} \\
&\qquad \times \|\omega(t)\|_{L^2(B(x_0,2r))} \,\|\psi\omega(t)\|_{L^2(B(x_0,2r))}\,dt \\
&\le \frac{1}{4} \sup_{t\in(t_0-(2r)^2,t_0)} ||\varphi\omega(t)||^2_{L^2(B(x_0,2r))} \\
&\qquad + c_2(r,\rho)\Big(\int ||\nabla\omega(t)||^2_{L^1(B(x_0,2r))}\,dt\Big)\Big(\int ||\omega(t)||^2_{L^2(B(x_0,2r))}\,dt\Big) \\
&\le \frac{1}{4} \sup_{t\in(t_0-(2r)^2,t_0)} ||\varphi\omega(t)||^2_{L^2(B(x_0,2r))} \\
&\qquad + c_2(r,\rho)\Big(\int (1+\|\nabla\omega(t)\|_{L^q(B(x_0,2r))})^{2q/(3q-3)}\,dt\Big) \\
&\qquad \times \Big(\int ||\omega(t)||^2_{L^2(B(x_0,2r))}\,dt\Big)
\end{aligned}$$

whenever

$$\frac{2q}{3q-3} \ge 2,$$

i.e., q in $(1, \frac{3}{2}]$.

Collecting all the estimates yields, for q in $(1, \frac{3}{2})$,

$$\frac{1}{2}\sup_{t\in(t_0-(2r)^2,t_0)}||\varphi\omega(t)||^2_{L^2(B(x_0,2r))}+||\nabla(\psi\omega)||^2_{L^2(Q_{2r})}$$
$$\leq\frac{1}{2}||\nabla(\psi\omega)||^2_{L^2(Q_{2r})}+\max\left\{c\|\nabla u\|_{L^2(Q_{2r})},\frac{1}{4}\right\}$$
$$\times\left(\frac{1}{2}\sup_{t\in(t_0-(2r)^2,t_0)}||\varphi\omega(t)||^2_{L^2(B(x_0,2r))}+||\nabla(\psi\omega)||^2_{L^2(Q_{2r})}\right)$$
$$+c\left(\int_{t_0-(2r)^2}^{t_0}||\nabla\omega(t)||^{2q/(3q-3)}_{L^q(B(x_0,2r))}\right)^{(3q-3)/(2q)}$$
$$\times\left(\frac{1}{2}\sup_{t\in(t_0-(2r)^2,t_0)}||\varphi\omega(t)||^2_{L^2(B(x_0,2r))}+||\nabla(\psi\omega)||^2_{L^2(Q_{2r})}\right)$$
$$+\frac{1}{8}\sup_{t\in(t_0-(2r)^2,t_0)}||\varphi\omega(t)||^2_{L^2(B(x_0,2r))}+M(r,\text{bounded quantities}).$$

Since u is a weak solution, there exists $d_1^*>0$ such that

$$c\|\nabla u\|_{L^2(Q_{2\delta})}\leq\frac{1}{4}$$

for all $\delta\leq d_1^*$. By the assumption, $\|\nabla\omega\|^{2q/(3q-3)}_{L^q(B(x_0,2R))}\in L^1(t_0-(2R)^2,t_0)$; hence, there exists $d_2^*>0$ such that

$$c\left(\int_{t_0-(2\delta)^2}^{t_0}||\nabla\omega(t)||^{2q/(3q-3)}_{L^q(B(x_0,2\delta))}\right)^{(3q-3)/(2q)}\leq\frac{1}{4}$$

for all $\delta\leq d_2^*$.

Let d^* be the minimum between d_1^* and d_2^*. If $r\leq d^*$, the nonlinearity is absorbed. If $r>d^*$, we can cover $B(x_0,2r)$ with finitely many balls $B(x_k,2d^*)$ and repeat the argument on each cylinder $B(x_k,2d^*)\times(t_0-(2d^*)^2,t_0)$.

Consider now a small positive ε.

For I_1, writing $1+1/r=(2+\varepsilon)/3+1/q_\varepsilon=\frac{2}{3}+1/q$ and renaming q_ε q, it transpires that the boundedness condition required is

$$||\nabla\omega||^{2q/((3-\varepsilon)q-3)}_{L^q(B(x_0,2R))}\in L^1(t_0-(2R)^2,t_0)\quad\text{for some } q,\ \frac{3}{3-\varepsilon}<q<\frac{3}{2-\varepsilon}.$$

For A, there is plenty of extra room between $1/|x-z|^{1+2\varepsilon}$ and $1/|x-z|^2$; hence no additional restrictions are needed.

For B, there is some extra room in the space-time powers in the estimate on

$$\int\|\nabla\omega(t)\|^2_{L^1(B(x_0,2r))}\,\mathrm{d}t;$$

however, it also depends on ε. It is possible to modify the estimates by further refining the cut-off function φ in order to have a good control on the higher order derivatives and use the Mean Value Theorem inductively to close the estimates. This is quite laborious. Instead, let us simply restrict q to the interval $(3/(3-\varepsilon), 3\mu/(2-\mu\varepsilon))$ for a μ, $0<\mu<1$, μ close to 1. This provides extra space in the space-time powers that is independent of ε and the singular kernel can readily be absorbed for small ε, $0<\varepsilon\le\varepsilon^*(\mu)$. □

Remark 3.3. If u is a suitable weak solution constructed in [3], the lower order term B can almost be estimated by *a priori* bounded quantities.

Let β be in $[\frac{4}{5},1)$, $\tau>0$, γ in $(0,\frac{1}{2}]$.

$$\int |\nabla\omega|^\beta \,\mathrm{d}x \le \left(\int |\nabla\omega|^2 \frac{1}{(1+\tau^2|\omega|^2)^{(1+\gamma)/2}}\,\mathrm{d}x\right)^{\beta/2} \times \left(\int (1+\tau^2|\omega|^2)^{((1+\gamma)/2)(\beta/(2-\beta))}\,\mathrm{d}x\right)^{(2-\beta)/2}.$$

Choosing γ such that $((1+\gamma)/2)(\beta/(2-\beta))=\frac{1}{2}$, raising both sides to the power of $2/\beta$ and integrating in time yields

$$\int\left[\int |\nabla\omega|^\beta\,\mathrm{d}x\right]^{2/\beta}\mathrm{d}t \le \left(\sup_t \int (1+\tau^2|\omega|^2)^{1/2}\,\mathrm{d}x\right)^{3/2} \times \iint |\nabla\omega|^2\frac{1}{(1+\tau^2|\omega|^2)^{(1+\gamma)/2}}\,\mathrm{d}x\,\mathrm{d}t$$

and this is finite for any $\gamma=2(1-\beta)/\beta$ in $(0,\frac{1}{2}]$ (cf. [3]).

Remark 3.4. For $q=\frac{4}{3}$, the boundedness condition needed to prevent the formation of singularities is

$$\|\nabla\omega\|_{4/3}^{8/(3-4\varepsilon)} \in L_t^1;$$

on the other hand, the *a priori* bound obtained in [3] is

$$\|\nabla\omega\|_{4/(3+\gamma)}^{4/(3+\gamma)} \in L_t^1.$$

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