Smoothness of Weak Solutions to a Nonlinear Fluid-structure Interaction Model

VIOREL BARBU, ZORAN GRUJIĆ, IRENA LASIECKA, & AMJAD TUFFAHA

ABSTRACT. The nonlinear fluid-structure interaction coupling the Navier-Stokes equations with a dynamic system of elasticity is considered. The coupling takes place on the boundary (interface) via the continuity of the normal component of the Cauchy stress tensor. Due to a mismatch of parabolic and hyperbolic regularity, previous results in the literature dealt with either a regularized version of the model, or with very smooth initial conditions leading to local existence only. In contrast, in the case of small but rapid oscillations of the interface, in [3] the authors established existence of *finite energy* weak solutions that are defined globally. This is achieved by exploiting new hyperbolic trace regularity results which provide a way to deal with the mismatch of parabolic and hyperbolic regularity. The goal of this paper is to establish regularity of *weak solutions*, for initial data satisfying the appropriate regularity and compatibility conditions imposed on the interface. It is shown that weak solutions equipped with smooth initial data become classical.

1. INTRODUCTION

1.1. The model. Interaction of a fluid/plasma and an elastic structure via an interface is a basic coupling in continuum mechanics. There are essentially two different scenarios: one in which an elastic solid is fully immersed in a fluid (e.g., a submarine submerged in an ocean or a microbubble suspended in a body fluid—used as a contrast in ultrasound imaging [19]), or adherence/detachment of leukocytes in a blood flow), and the other one is when a fluid is flowing along a pipe or is filling a container with elastic walls (e.g. [17, 18, 20]), or the flow of blood along blood vessels [7]. We focus on the first case. However, mathematical subtleties are common to both cases. While there has been a lot of interest and

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attention paid to understanding the dynamics of these structures, the vast majority of papers is devoted to numerical and experimental studies. Mathematical, PDE-oriented analysis, instead, is rather scarce, with many problems still unresolved. This, in particular, refers to mathematically fundamental issues such as well-posedness of weak solutions and their regularity. The latter is the main topic of the present paper.

In what follows we describe the model under consideration. Let $\Omega \subset R^n$, $n = 2, 3$, be a bounded simply connected domain with an interior region Ω_s (a domain occupied by an elastic solid) and an exterior region Ω_f (a domain filled with viscous incompressible fluid). Denote by Γ_f the outer boundary of the domain Ω_f, and by Γ_s the boundary of the region Ω_s which is also an interior boundary of Ω_f, and where the interaction takes place. Let u be a vector-valued function defined on $\Omega_f \times [0, T]$ representing the velocity of the fluid, and p a scalar-valued function representing the pressure. Additionally, let w, w_t be the displacement and the velocity functions of the elastic solid Ω_s. We also denote by ν the unit outward normal vector on Γ_s with respect to the region Ω_s.

We are considering the following PDE model of fluid-structure interaction defined by the variables (u, w, w_t, p).

$$
(1.1) \qquad \begin{cases}
u_t - \operatorname{div}\varepsilon(u) + (u \cdot \nabla)u + \nabla p = 0, & \Omega_f \times (0,T), \\
\operatorname{div} u = 0, & \Omega_f \times (0,T), \\
w_{tt} - \operatorname{div}\sigma(w) = 0, & \Omega_s \times (0,T), \\
u(0,\cdot) = u_0, & \Omega_f, \\
w(0,\cdot) = w_0, \; w_t(0,\cdot) = w_1, & \Omega_s, \\
u = 0, & \Gamma_f \times (0,T), \\
w_t = u, & \Gamma_s \times (0,T), \\
\sigma(w) \cdot \nu = \varepsilon(u) \cdot \nu - p\nu + \dfrac{1}{2}(u \cdot \nu)u, & \Gamma_s \times (0,T),
\end{cases}
$$

where the elastic stress tensor σ and the strain tensor, respectively, are given by

$$
\sigma_{ij}(u) = \lambda \sum_{k=1}^{k=3} \varepsilon_{kk}(u)\delta_{ij} + 2\mu\varepsilon_{ij}(u), \quad \lambda,\ \mu > 0,
$$

$$
\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).
$$

Note that the continuity of both the velocities and the normal components of the stress tensors across the interface is required.

In the system considered, the interface Γ_s is stationary. This corresponds to a physical situation in which the order of magnitude of the displacement of the elastic solid on the boundary is smaller than the order of magnitude of the velocity (small but rapid oscillations) [13, 29]. If one considers the case of a moving interface, the equation for the interface in the Lagrangian coordinates is comparable to our elastic equation [11], and the core of the problem arising from the parabolic-hyperbolic coupling across the interface via the continuity of the velocities and the normal components of the stress tensors is similar.

Also, note that the presence of the—not necessarily small—fluid term $\frac{1}{2}(u \cdot \nu)u$ on Γ_s is due to the fact that the interface is stationary. Namely, in addition to the normal component of the fluid Cauchy stress tensor $\mathcal{T} = -pI + \varepsilon(u)$, where ε is the deformation tensor, which would be present in the case of the moving interface as well, this model features an additional stress exerted on the interface originating in the tendency of the fluid to advect through the interface. The advection term in the Navier-Stokes equations is $(u \cdot \nabla)u$, which is, due to the incompressibility of the fluid, equal to div $(u \otimes u)$, and which corresponds exactly to a boundary term of the $(u \otimes u)\nu = (u \cdot \nu)u$-type (see also [29]). In the case of a moving interface [11], this boundary term is entirely absorbed by the material derivative of the velocity of the fluid.

This model (with both moving and stationary interface) has been well-established in both mathematical and modeling literature—see, e.g., [11, 13–15, 27, 31, 32]—and the applications range from naval and aerospace engineering to cell biology and biomedical engineering [17, 18, 20]. However, due to the nature of this particular form of parabolic-hyperbolic coupling, even the basic question of existence of the natural energy-class weak solutions had not been previously resolved. A key issue is that the traces of the elastic (wave/hyperbolic) component at the energy level are not defined via the standard trace theory.

There had been two different avenues—effective also in the case of moving interface—in approaching this problem. The first one had been to add a "structural damping" term, thus effectively *regularizing* the elastic/hyperbolic dynamics (cf. [8, 9] and the references therein) and clearing the stage for the standard trace theory to apply. In this approach, the heart of the difficulty related to boundary traces is essentially defined away. The other avenue had been to consider the case of *very smooth* data—this led to a functional setting in which the standard trace theory applies yielding local-in-time existence of smooth solutions (cf. [11, 14]). However, when dealing with the original non-regularized model within the framework of finite energy solutions, with moving or stationary interface, the main mathematical obstacles remain to be:

- Mismatch between parabolic and hyperbolic regularity, which is most pronounced on the interface where the traces of hyperbolic solutions are not *a priori* defined in the topology of finite energy space.
- The presence of Neumann type boundary conditions that rules out standard approaches to the NSE equations via Leray's projection.

- The coupling between fluid and the structure taking place on the interface-boundary, hence contributing to the issue of mismatch of regularity between the two types of dynamics.

In a very recent work [3], the issues raised above in the case of stationary interface have been successfully dealt with and the authors established global-in-time existence of the energy-level weak solutions to (1.1) *without any* regularization of the elastic/hyperbolic dynamics. One of the key ingredients was establishment of an *improved* (often referred as "hidden" [28]) trace regularity of hyperbolic solutions, that provides a way to deal with the mismatch of the regularity. As a consequence, the functional spaces for the fluid component are exactly the same as in the classical Leray theory of weak solutions for the NSE *per se*. The proof was based on an interplay among nonlinear semigroup, variational and weak-compactness methods, starting from an approximate problem defined via a suitable truncation of the NSE nonlinearity, and enhanced by a careful micro-local analysis argument revealing "hidden boundary regularity" properties of the elastic/hyperbolic component. In what follows we shall recall, for the reader's convenience, some of these results.

1.2. Notation. Throughout the paper $\mathcal{H} \equiv H \times H^1(\Omega_s) \times L_2(\Omega_s)$ where

$$H \equiv \{u \in L_2(\Omega_f) : \operatorname{div} u = 0,\ u \cdot \nu|_{\Gamma_f} = 0\}$$

will denote the energy space for the system.

Note that all Sobolev spaces H^s, L_2 pertaining to u and w are in fact $(H^s)^n$, $(L_2)^n$, $n = 2, 3$, and only for simplicity we omit the exponent n.

In addition we will use the following notation:

$$V \equiv \{v \in H^1(\Omega_f) : \operatorname{div} v = 0,\ u|_{\Gamma_f} = 0\}$$

$$(u, v)_f = \int_{\Omega_f} uv \, d\Omega_f; \quad (u, v)_s = \int_{\Omega_s} uv \, d\Omega_s;$$

$$\langle u, v \rangle = \int_{\Gamma_s} uv \, d\Gamma_s; \quad D_i = \frac{\partial}{\partial x_i}$$

$$|u|_{s,D} = |u|_{H^s(D)}; \quad |u|_s = |u|_{s,\Omega}; \quad |u| = |u|_{0,\Omega}.$$

V is topologized with respect to the inner product given by:

$$(u, v)_{1,f} \equiv \int_{\Omega_f} \varepsilon(u)\varepsilon(v) \, d\Omega_f$$

We denote the induced norm by $|\cdot|_{1,\Omega_f}$ which is a norm equivalent to the usual $H^1(\Omega_f)$ norm via Korn's inequality and Poincare's inequality

$$|u|_{1,\Omega_f} = \left[\int_{\Omega_f} |\varepsilon(u)|^2 \, d\Omega_f \right]^{1/2}.$$

$H^1(\Omega_s)$ is topologized with respect to the inner product given by

$$(w,z)_{1,s} \equiv \int_{\Omega_s} wz + \int_{\Omega_s} \sigma(w)\varepsilon(z).$$

We denote by $|\cdot|_{1,\Omega_s}$ the induced norm by the inner product above

$$|w|^2_{1,\Omega_s} = \left[\int_{\Omega_s} \sigma(w)\varepsilon(w)\,\mathrm{d}\Omega_s\right] + |w|^2_{0,\Omega_s}.$$

The norm $|\cdot|_{1,\Omega_s}$ is again equivalent to the usual $H^1(\Omega_s)$ norm by Korn's inequality. The boundary integrals are defined as $\langle u,v\rangle_{\Gamma_s} = \int_{\Gamma_s} uv\,\mathrm{d}s$.

By $C_w([0,T],X)$ we denote a space of functions that are weakly continuous in t with values in X.

C will denote a generic constant, which can be different at different occurrences.

Finally, the energy functional for the system is given by

$$E(t) \equiv |u(t)|^2_{0,\Omega_f} + \Big(\sigma(w(t)),\varepsilon(w(t))\Big)_s + |w_t(t)|^2_{0,\Omega_s}$$

.

1.3. Known results. We begin by defining weak solutions to the original system system (1.1) . This is obtained by projecting the equations to H and utilizing the boundary conditions.

Definition 1.1 (Weak solution)**.** Let $(u_0,w_0,w_1) \in \mathcal{H}$ and $T>0$. We say that a triple $(u,w,w_t) \in C_w([0,T];H\times H^1(\Omega_s)\times L_2(\Omega_s)) = C_w([0,T];\mathcal{H})$, $u \in L_2((0,T);V)$ is a weak solution of (1.1) if

- $(u(0),w(0),w_t(0)) = (u_0,w_0,w_1)$ (in the sense of weak continuity),
- $\sigma(w)\cdot\nu \in L_2((0,T);H^{-1/2}(\Gamma_s))$,
- $\frac{\mathrm{d}}{\mathrm{d}t} w|_{\Gamma_s} = u|_{\Gamma_s} \in L_2((0,T),H^{1/2}(\Gamma_s))$, and the following variational system holds a.e. in $t\in(0,T)$:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(u_t,\varphi)_f + (\varepsilon(u),\varepsilon(\varphi))_f + ((u\cdot\nabla)u,\varphi)_f + \langle\sigma(w)\cdot\nu,\varphi\rangle \\ \qquad\qquad - \frac{1}{2}\langle(u\cdot\nu)u,\varphi\rangle = 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}(w_t,\psi)_s + (\sigma(w),\varepsilon(\psi))_s - \langle\sigma(w)\cdot\nu,\psi\rangle = 0 \end{cases} \tag{1.2}$$

for all test functions $\varphi \in V$ and $\psi \in H^1(\Omega_s)$.

Remark 1.2.

1. Note that weak solutions, as defined in Definition 1.1, require information on the trace $\sigma(w) \cdot \nu|_{\Gamma_s}$. This does not follow from the interior regularity of finite energy solutions. Thus, the definition of *weak* solutions imposes *an additional regularity* requirement on the normal stress of the solid's displacement w on the interface. The fact that such requirement is necessary, follows from the variational principle used with independent test functions φ, ψ which are not required to match on the interface [31], hence they retain normal stresses in the formulation. In short, the presence of $\sigma(w) \cdot \nu$ in the boundary terms is an intrinsic feature of the definition of weak solutions which, in turn, requires imposition of the trace regularity postulated by Definition 1.3. The key point we want to make is that this additional regularity (which does not follow from any trace theory) is shown to be a *property* of finite energy solutions, rather than an artifact assumed arbitrarily on solution. This will be documented below.
2. The time derivative of the trace of w, appearing in the condition expressing matching of velocities, is understood in the sense of distribution. The intrinsic regularity of the fluid on the boundary allows to identify the distribution with $L_2(0, T, H^{1/2}(\Gamma_s))$ function.

A starting point of our analysis is existence result for weak and global solutions obtained in [3].

Theorem 1.3 (Existence of weak solutions)**.** *Given any initial condition* $(u_0, w_0, w_1) \in \mathcal{H}$ *and any* $T > 0$*, there exists a weak solution* (u, w, w_t) *to the system* (1.1) *such that*

$$\nabla w \Big|_{\Gamma_s} \in L_2((0,T); H^{-1/2}(\Gamma_s)),$$

$$\frac{\mathrm{d}}{\mathrm{d}t} w \Big|_{\Gamma_s} = w_t \Big|_{\Gamma_s} \in L_2((0,T); H^{1/2}(\Gamma_s)).$$

Moreover, in the case when dimension of $\Omega = 2$*, weak solutions are unique within the class specified above.*

As mentioned before, a weak solution as defined in Definition 1.1 requires information on the trace $\sigma(w) \cdot \nu|_{\Gamma_s}$, which does not follow from finite energy regularity of solutions. Fortunately, Theorem 1.3 does provide existence of finite energy solutions with the *additional* boundary regularity. This confirms that the definition of *weak* solution is a correct one for the problem under consideration.

The key result used in order to establish additional boundary regularity in (1.3) is the following trace regularity of finite energy solutions to a linear elastic wave equation.

Lemma 1.4 ([3]). *Let (w, w_t) be a solution to an elastic wave equation defined distributionally on $\Omega \times (0, T)$*

$$w_{tt} - \operatorname{div} \sigma(w) = 0$$

driven by the folllowing data:

$$w(0) \in H^1(\Omega_s), \quad w_t(0) \in L_2(\Omega_s), \quad \frac{\mathrm{d}}{\mathrm{d}t} w \bigg|_{\Gamma_s} \in L_2(0, T; H^{1/2}(\Gamma_s)).$$

Then $(\partial/\partial\nu)w \in L_2((0, T) \times \Gamma_s) \oplus C([0, T]; H^{-1/2}(\Gamma_s))$.

Remark 1.5. We note that the trace result stated in Lemma 1.4 does not follow from the interior regularity of solutions to the wave equation. It is an independent regularity result, inspired by techniques developed in [6, 22, 23, 28], and obtained by microlocalizing the problem to "hyperbolic" and "elliptic" sectors that represent $L_2((0, T) \times \Gamma_s)$ and $C([0, T]; H^{-1/2}(\Gamma_s))$ regularity of the normal derivatives, respectively.

The goal of this paper is to show that if the initial data are sufficiently smooth and satisfy natural compatibility conditions on the interface Γ_s, the weak solutions obtained in Theorem 1.3 are in fact smooth. The main results will be stated in the following section, and the rest of the paper is devoted to the proofs.

1.4. New results. Strong solutions refer to the original PDE system and they are defined as follows:

Definition 1.6 (Strong solutions). We say that (u, w, w_t, p) is a strong solution of (1.1) if

- $(u, w, w_t) \in L_\infty((0, T); V \times H^2(\Omega_s) \times H^1(\Omega_s))$.
- $(u, p) \in L_2((0, T); H^2(\Omega_f) \times H^1(\Omega_f))$.
- $(u_t, w_t, w_{tt}) \in L_\infty((0, T); H \times H^1(\Omega_s) \times L_2(\Omega_s)) = L_\infty((0, T); \mathcal{H})$.
- The strong form of equations given in (1.1) holds a.e. in $\Omega \times (0, T)$.

As expected, in order to be able to obtain strong solutions, one must impose a suitable compatibility conditions on the initial data. These are formulated below.

Definition 1.7 (Compatibility Conditions (CC)). We say that initial conditions $(u_0, w_0, w_1) \in H^2(\Omega_f) \cap V \times H^2(\Omega_s) \times H^1(\Omega_s)$ satisfy Compatibility Conditions (CC) if

- $w_1 = u_0,\ on\ \Gamma_s$
- $\langle \sigma(w_0) \cdot \nu - \varepsilon(u_0) \cdot \nu + 1/2(u \cdot \nu)u, \varphi \rangle = 0$ for all $\varphi \in V$.

In order to formulate our results, we shall distinguish two and three dimensional domains.

1.4.1. The two dimensional case

Theorem 1.8. *Let* $\Omega \subset R^2$. *Then given* $(u_0, w_0, w_1) \in H^2(\Omega_f) \bigcap V \times H^2(\Omega_s) \times H^1(\Omega_s)$ *such that the compatibility conditions* (CC) *are satisfied, the following holds. A weak solution asserted by Theorem 1.3 becomes a strong solution* (u, w, w_t, p) *satisfying the system* (1.1). *Moreover,* $u \in C([0,T];V)$ *and the strong solutions are unique.*

Remark 1.9. We note that the H^2 regularity of fluid component is only L_2 in time, rather than L_∞—as in the classical Navier-Stokes equations. This is due to a topological mismatch between the fluid and the solid—a feature that characterizes the interaction.

1.4.2. The three dimensional case. In the three-dimensional case we shall consider two different situations, i.e., local-in-time strong solutions for arbitrary large initial data and global-in-time strong solutions for small initial data.

- Local-in-time strong solutions for general initial data

Theorem 1.10. *Let* $\Omega \subset R^3$. *Then given* $(u_0, w_0, w_1) \in H^2(\Omega_f) \bigcap V \times H^2(\Omega_s) \times H^1(\Omega_s)$ *such that the compatibility conditions* (CC) *are satisfied, there exists* $T' > 0$ *such that a weak solution on* $(0, T')$ *becomes a strong solution* (u, w, w_t, p) *satisfying the system* (1.1). *Moreover,* $u \in C([0,T'];V)$ *and the strong solutions are unique.*

- Global-in-time strong solutions for small initial data

Theorem 1.11. *Let* $\Omega \subset R^3$, *and* $(u_0, w_0, w_1) \in H^2(\Omega_f) \bigcap V \times H^2(\Omega_s) \times H^1(\Omega_s)$ *such that the compatibility conditions* (CC) *are satisfied. Assume that*

$$|u_0|^2_{2,\Omega_f} + |w_0|^2_{2,\Omega_s} + |w_1|^2_{1,\Omega_s} \le C$$

for a suitable absolute constant C. *Then, there exists a unique strong solution* (u, w, w_t, p), $u \in C([0,\infty);V)$, *satisfying the system* (1.1) *on* $(0,\infty)$.

Strong solutions correspond to the original PDE—hence they involve the pressure term p. Having proved existence of weak solutions in [3], the next step is to analyze regularity of these solutions given smooth initial data satisfying the compatibility conditions (CC). The proof of regularity/smoothness relies on the following major steps:

Step 1: We prove that *time derivatives* are also bounded in the finite energy space $\mathcal{H}$.

Step 2: Time regularity of weak solutions allows for a reconstruction of the PDE form and, in particular, for the identification of the pressure.

Step 3: In this step, we aim at obtaining higher space regularity. This step consists of two sub-steps. First we prove the additional regularity of the tangential derivatives of solutions defined in a collar neighborhood of the interface Γ_s. In the second step we reconstruct full H^2 regularity of solutions by appealing to a version of Agmon-Douglis-Nirenberg methods. Thus, at the end of the process we obtain regular (classical) solutions corresponding to the original problem equipped with"smooth" and compatible initial conditions.

Remark 1.12. The results presented above pertain to *static* interface model. However, it is our belief that the techniques developed here should have strong bearing on the theory of weak and strong well-posedness for the corresponding *moving* interface problem which, to the best of our knowledge, is an open and very challenging issue.

2. PDE Solutions to a Nonhomogeneous Linear Problem

2.1. Preliminaries: Characterization of $H^{\perp}$–the orthogonal complement of H in $L_2(\Omega_f)$. The following regularity results are known. We include the arguments for the sake of completeness.

Lemma 2.1. *Suppose $p \in H^{1/2}(\Gamma_s)$ and $\langle p, g\rangle_{\Gamma_s} = 0$ for all $g \in H^{-1/2}(\Gamma_s)$ satisfying $\int_{\Gamma_s} g = 0$. Then p is constant on Γ_s.*

Proof. Let $g = p - c$ where $c = (1/m(\Gamma_s)) \int_{\Gamma_s} p$ is a constant. Then $g \in H^{1/2}(\Gamma_s)$ and $\int_{\Gamma_s} g = 0$. By the assumption, $\langle p, g\rangle_{\Gamma_s} = \int_{\Gamma_s} p(p - c) = 0$, and $c\int_{\Gamma_s} g = c\int_{\Gamma_s}(p-c) = 0$. Therefore $0 = \int_{\Gamma_s} p(p-c) - \int_{\Gamma_s} c(p-c) = \int_{\Gamma_s}(p-c)^2$. Hence $p = c$ a.e. on Γ_s. ◻

Corollary 2.2. *Let $p \in H^{\frac{1}{2}}(\Gamma_s)$ and suppose $\langle p, \varphi \cdot \nu\rangle_{\Gamma_s} = 0$ for all $\varphi \in H$. Then p is constant on Γ_s*

Proof. Let y_0 be defined by $y_0 = \nabla q$ where $q \in H^1(\Omega_f)$ is a solution to the Neumann problem $\Delta q = 0$, $(\partial/\partial \nu)q\big|_{\Gamma_s} = g$ and $(\partial/\partial \nu)q\big|_{\Gamma_f} = 0$ for some $g \in H^{-1/2}(\Gamma_s)$ satisfying the compatibility condition $\int_{\Gamma_s} g = 0$.

Note that $y_0 \cdot \nu\big|_{\Gamma_s} = (\partial/\partial\nu)q|_{\Gamma_s} = g$. Since $y_0 \in H$, by assumption, $\langle p, y_0 \cdot \nu\rangle_{\Gamma_s} = 0$ and thus $\langle p, g\rangle_{\Gamma_s} = 0$ for all $g \in H^{-1/2}(\Omega_f)$ satisfying the condition $\int_{\Gamma_s} g = 0$. Therefore, Lemma 2.1 applies and we conclude that p is constant on Γ_s. ◻

Theorem 2.3.

$$H^{\perp} = \{v : v = \nabla p,\ p \in H^1(\Omega_f),\ p|_{\Gamma_s} = \text{constant}\}. \tag{2.1}$$

Proof. We first denote by H_0 the space

$$\{v \in L_2(\Omega_f) : \operatorname{div} v = 0,\ v \cdot \nu|_\Gamma = 0\} \quad (\Gamma = \Gamma_s \cup \Gamma_f)$$

whose orthogonal complement $H_0^\perp$ is well known to be

$$\{v : v = \nabla p,\ p \in H^1(\Omega_f)\}$$

(see, e.g., [10, 33]). Clearly, $H_0 \subset H$. Let $\eta \in H^\perp$, denote by P_{H_0} the projection onto H_0, and let $x = P_{H_0}\eta$. Then $\eta = x + \nabla p$ for some $p \in H^1(\Omega_f)$. Also, for all $y \in H$, $(y, \eta) = 0$ and hence

$$(x, y) + (\nabla p, y) = 0.$$

Applying the divergence theorem, and using the fact that $\operatorname{div} y = 0$ since $y \in H$, we obtain

$$(x, y) + \langle p,\ y \cdot \nu \rangle_{\Gamma_s} = 0.$$

Since $H_0 \subset H$, $(\eta, z) = 0$ for all $z \in H_0$ yielding $\eta \in H_0^\perp$, i.e., $x = 0$. Therefore

$$\langle p, y \cdot \nu \rangle_{\Gamma_s} = 0 \tag{2.2}$$

for all $y \in H$. Thus, by Corollary 2.2 $p|_{\Gamma_s}$ = constant, concluding that for any $\eta \in H^\perp, \eta = \nabla p$ where $p \in H^1(\Omega_f)$ and p = constant on Γ_s.

Conversely, if $\eta = \nabla p$ with $p \in H^1(\Omega_f)$, p = constant on Γ_s, then for all $y \in H$,

$$(\eta, y) = (\nabla p, y) = \langle p, y \cdot \nu \rangle_{\Gamma_s} = \langle c, (y \cdot \nu) \rangle = c \int_\Gamma y \cdot \nu = c \int_{\Omega_s} \operatorname{div} y = 0.$$

Hence, $\eta \in H^\perp$. □

2.2. PDE solutions to the non-homogenous linear problem. In this section we shall study existence of strong solutions to the following linear problem. Let

$$\mathcal{T}(u, p) = -pI + \varepsilon(u)$$

be the fluid Cauchy stress tensor and the space

$$X \equiv \{\varphi|_{\Gamma_s} : \varphi \in V\} = \{z \in H^{1/2}(\Gamma_s) : \int_{\Gamma_s} z \cdot \nu = 0\}. \tag{2.3}$$

With $f \in L_2((0, T); L_q(\Omega_f) \cap V')$, for some $q \geq 1$ and $g \in L_2([0, T]; X')$, we consider

$$
(2.4)\qquad \begin{cases} u_t - \operatorname{div} \mathcal{T}(u,p) + f = 0, & \Omega_f \times (0,T)\,, \\ \operatorname{div} u = 0, & \Omega_f \times (0,T)\,, \\ w_{tt} - \operatorname{div}\ \sigma(w) = 0, & \Omega_s \times (0,T)\,, \\ u(0,\cdot) = u_0, & \Omega_f\,, \\ w(0,\cdot) = w_0,\ w_t(0,\cdot) = w_1, & \Omega_s\,, \\ w_t = u, & \Gamma_s \times (0,T)\,, \\ u = 0, & \Gamma_f \times (0,T)\,, \\ \sigma(w)\cdot\nu = \mathcal{T}(u,p)\cdot\nu - g, & \text{in } L_2((0,T);X') \end{cases}
$$

Then we obtain the following result:

Lemma 2.4. *Let $Y = (u, w, w_t)$ be a weak solution corresponding to (2.4) such that $(d/dt)Y \in L_\infty((0,T);\mathcal{H})$. Let $q^* \equiv \min\{2,q\}$. Then, there exists $p \in L_2((0,T);L_2(\Omega_f))$ such that*

$$
\begin{aligned}
\operatorname{div}\mathcal{T}(u,p) &\in L_2((0,T),L_{q^*}(\Omega_f))\,,\\
\mathcal{T}(u,p)\cdot\nu &\in L_2(0,T;H^{-1/2}(\Gamma_s))\,,\\
\operatorname{div}\sigma(w) &\in L_\infty((0,T),L_2(\Omega_s))\,,
\end{aligned}
$$

and (u, w, w_t, p) satisfies (2.4).

Proof. Our starting point is the following variational formulation for (u,w) with $\varphi \in V$ and $\psi \in H^1(\Omega_s)$ analogous to (1.2) in definition of weak solutions (1.1):

$$
(2.5)\qquad (\varepsilon(u),\varepsilon(\varphi))_f + (f,\varphi)_f + \langle \sigma(w)\cdot\nu + g,\varphi\rangle = -(u_t,\varphi)_f\,,
$$

$$
(2.6)\qquad w_t\Big|_{\Gamma_s} = u\Big|_{\Gamma_s} \quad \text{in } L_2([0,T],H^{1/2}(\Gamma_s))\,,
$$

$$
(2.7)\qquad (\sigma(w),\varepsilon(\psi))_s - \langle \sigma(w)\cdot\nu,\psi\rangle = -(w_{tt},\psi)_s\,,
$$

$\forall \varphi \in V$, $\psi \in H^1(\Omega_s)$ a.e. in $(0,T)$. Here we took advantage of assumed additional regularity of Y_t.

Taking $\varphi \in H^1_0(\Omega_f)\cap V$, $\psi \in H^1_0(\Omega_s)$ and accounting for the fact that $u_t(t) \in H$ and $w_{tt}(t)$ is $\in L_2(\Omega_s)$, we infer from (2.5) and (2.7)

$$
\Big(-\operatorname{div}\ \varepsilon(u(t)) + f(t) + u_t(t),\varphi\Big)_f = 0,
$$
$$
\forall \varphi \in V_0 \equiv H^1_0(\Omega_s)\cap H,\ \text{ a.e. in } t \in (0,T)
$$

and

$$w_{tt}(t) \; - \; \operatorname{div} \sigma(w(t)) \; = \; 0, \quad \text{in } H^{-1}(\Omega_s), \quad \text{a.e. in } t \; \in \; (0,T)\,.$$

Since, in particular, $f + u_t \in V'$, *a.e.* in $(0,T)$, de Rham's theorem implies that

$$-\operatorname{div} \mathcal{T}(u,p) + f + u_t = 0$$

in the sense of distributions for some $p \in L_2((0,T);L_2(\Omega_f))$. Since we also have that $u_t + f \in L_2((0,T);L_{q^*}(\Omega_f))$ and w_{tt} is $\in L_\infty((0,T);L_2(\Omega_s))$, we infer the strong form of the equations

$$\begin{aligned} -\operatorname{div} \mathcal{T}(u,p) + f + u_t &= 0, \quad \text{in } L_2((0,T),L_{q^*}(\Omega_f))\,, \\ w_{tt} - \operatorname{div} \sigma(w) &= 0, \quad \text{in } L_\infty((0,T),L_2(\Omega_s))\,. \end{aligned} \tag{2.8}$$

Since

$$\sigma(w) \cdot \nu \in L_2((0,T);\, H^{-1/2}(\Gamma_s)) \tag{2.9}$$

(hidden regularity of *weak solution* postulated by the Definition 1.1 and implied by Theorem 1.3), utilizing the weak form of Green's formula along with (2.5), (2.8), we obtain

$$\langle \mathcal{T}(u,p)\cdot\nu + \sigma(w)\cdot\nu + g,\, \varphi\rangle = 0 \quad \forall \varphi \in V.$$

Since $\varphi|_{\Gamma_s} \in X$,

$$\mathcal{T}(u,p)\cdot\nu + \sigma(w)\cdot\nu + g \in L_2([0,T];[X]^\perp),$$

and since the normal cone to X (in L_2) is $\{k\nu\}$, $k \in \mathbb{R}$, we infer existence of some $\lambda \in L_2([0,T])$ such that

$$\mathcal{T}(u,p) \cdot \nu + \sigma(w) \cdot \nu + g = \lambda(t)\nu\,. \tag{2.10}$$

Note the choice of p is up to a constant so that $\lambda(t)$ can be taken equal zero. The hidden regularity of $\sigma(w) \cdot \nu$ (2.9) implies boundary regularity of the Cauchy stress tensor $\mathcal{T}(u,p) \cdot \nu \in L_2(0,T;H^{-1/2}(\Gamma_s))$. □

3. STRONG SOLUTIONS TO THE FULL NONLINEAR SYSTEM

3.1. Preliminaries. Consider now the full nonlinear system

$$
\left\{
\begin{array}{rlr}
u_t - \operatorname{div} \mathcal{T}(u,p) + (u\cdot\nabla)u = 0\,, & \Omega_f \times (0,T)\,, \\
\operatorname{div} u = 0\,, & \Omega_f \times (0,T)\,, \\
w_{tt} - \operatorname{div} \sigma(w) = 0\,, & \Omega_s \times (0,T)\,, \\
u(0,\cdot) = u_0\,, & \Omega_f\,, \\
w(0,\cdot) = w_0,\ w_t(0,\cdot) = w_1\,, & \Omega_s\,, \\
w_t = u\,, & \Gamma_s \times (0,T)\,, \\
u = 0\,, & \Gamma_f \times (0,T)\,, \\
\sigma(w)\cdot\nu = \mathcal{T}(u,p)\cdot\nu - p\nu + \frac{1}{2}(u\cdot\nu)u\,, & \Gamma_s \times (0,T)\,,
\end{array}
\right. \tag{3.1}
$$

Our first goal is to apply Lemma 2.4 to the system (3.1) with $f = (u\cdot\nabla)u$ and $g = -\frac{1}{2}(u\cdot\nu)u$.

Before we proceed, let us recall the notation and an estimate on the trilinear form corresponding to our system (cf. [3]). Denote by $B(\cdot,\cdot) : V \times V \to V'$ the operator defined by

$$(B(y,x),z)_f \equiv ((y\cdot\nabla)x, z)_f - \tfrac{1}{2}\langle (y\cdot\nu)x,\ z\rangle, \tag{3.2}$$

and by $B(\cdot) : V \to V'$ the operator defined by

$$B(u) \equiv B(u,u). \tag{3.3}$$

Also, denote by $b(u,v,w)$ the trilinear form

$$b(u,v,w) \equiv (B(u,v),w)_f = ((u\cdot\nabla)v,w)_f - \tfrac{1}{2}\langle (u\cdot\nu)v, w\rangle. \tag{3.4}$$

In what follows, we shall use the following notation for the H^s norm on Ω_f

$$|u|_s = |u|_{s,\Omega_f}, \quad |u|_0 = |u|_{L_2(\Omega_f)}.$$

Lemma 3.1. *The trilinear mapping b satisfies the following estimates for all $u \in V$, v, $w \in H^1(\Omega_f)$ (n is dimension of Ω):*

$$\left|((u\cdot\nabla)v,w)_f\right| \le C|u|_{s_1}\, |v|_{s_2+1}\, |w|_{s_3} \tag{1}$$

where s_1, s_2, $s_3 \ge 0$ and $s_1 + s_2 + s_3 \ge n/2$ if $s_i \ne n/2$ $\forall i = 1,2,3$ or $s_1 + s_2 + s_3 > n/2$ if $s_i = n/2$ for some $i = 1,2,3$.

(2) $$\left|\langle (u\cdot \nu)v, w\rangle\right| \le C|u|_{s_1}\,|v|_{s_2}\,|w|_{s_3}$$

where $s_1 \ge \frac{1}{2}$, $s_2, s_3 > \frac{1}{2}$ *and* $s_1 + s_2 + s_3 \ge (n+2)/2$.

(3) *If* $n = 2$

$$|b(u,v,w)| \le C_1|u|_{1/2}\,|v|_1\,|w|_{1/2} + C_2|u|_{1/2}\,|v|_{3/4}\,|w|_{3/4}\,.$$

(4) *If* $n = 3$,

$$|b(u,v,w)| \le C|u|_{1/2}\,|v|_1\,|w|_1\,,$$

$$|b(u,v,w)| \le C|u|_{3/4}\,|v|_1\,|w|_{3/4}\,.$$

(5) $b(u,v,w) = -b(u,w,v)$, $b(u,u,u) = b(u,v,v) = 0$.

(Item (1) above is the standard NSE trilinear estimate, while (2)-(5) were proved in [3].)

Lemma 3.2. *For* $n = 2,3$ *and* $u \in L_2([0,T];V)$, *we have that* $(u\cdot\nu)u \in L_1([0,T],X')$.

Proof. Let $\varphi \in L_\infty([0,T];X)$, then:

$$\begin{aligned}\int_0^T \langle (u(s,\cdot)\cdot\nu)u(s,\cdot),\varphi(s,\cdot)\rangle\,\mathrm{d}s &\le \int_0^T |u(s,\cdot)|^2_{L_4(\Gamma_s)}|\varphi(s,\cdot)|_{L_2(\Gamma_s)}\,\mathrm{d}s\\ &\le \int_0^T C|u(s,\cdot)|^2_{1/2,\Gamma_s}|\varphi(s,\cdot)|_{1/2,\Gamma_s}\,\mathrm{d}s\\ &\le C\int_0^T |u(s,\cdot)|^2_{1,\Omega_f}\,\mathrm{d}s\,|\varphi|_{L_\infty([0,T],X)}\,,\end{aligned}$$

where we used the Sobolev continuous imbedding of $H^{1/2}$ into L_4 on the boundary Γ_s in both one and two dimensions. The desired result now follows from taking the supremum over all $\varphi \in L_\infty([0,T];X)$ of the unit norm. □

Lemma 3.3. *Let* $n = 2,3$. *If the initial conditions*

$$u_0,\ w_0,\ w_1 \in H^2(\Omega_f)\cap V \times H^2(\Omega_s)\times H^1(\Omega_s)$$

satisfy the compatibility conditions (CC) *stated in* (1.7), *then a weak solution* u, w, w_t *satisfying* (1.2) *is such that* $(u_t(0), w_1, w_{tt}(0)) \in \mathcal{H}$.

Proof. Applying Green's formula in the weak formulation 1.2 gives

$$(u_t(t), \varphi)_f = -\Big(\varepsilon(u(t)), \varepsilon(\varphi)\Big)_f - \Big((u(t)\cdot\nabla)u(t), \varphi\Big)_f - \Big\langle \sigma(w(t))\cdot\nu, \varphi\Big\rangle + \Big\langle \tfrac{1}{2}(u(t)\cdot\nu)u(t), \varphi\Big\rangle$$

$$= \Big(\operatorname{div}\varepsilon(u(t)) - (u(t)\cdot\nabla)u(t), \varphi\Big)_f - \Big\langle [\sigma(w(t))\cdot\nu + \varepsilon(u(t))\cdot\nu + \tfrac{1}{2}(u(t)\cdot\nu)u(t)], \varphi\Big\rangle .$$

Taking the weak limit as $t \to 0+$ yields

$$(u_t(0), \varphi)_f = (\operatorname{div}\varepsilon(u_0) - (u_0\cdot\nabla)u_0, \varphi)_f - \Big\langle [\sigma(w_0)\cdot\nu + \varepsilon(u_0)\cdot\nu + \tfrac{1}{2}(u_0\cdot\nu)u_0], \varphi\Big\rangle.$$

Next, we use the compatibility condition in Definition 1.7 satisfied by the initial condition u_0, w_0, w_1 to obtain

$$(u_t(0), \varphi)_f = (\operatorname{div}\varepsilon(u_0) - (u_0\cdot\nabla)u_0, \varphi)_f$$

for all $\varphi \in V$. We extend the equation to all $\varphi \in H$ by density of V in H. This implies

$$|u_t(0)|_{0,\Omega_f} \le C|\operatorname{div}\varepsilon(u_0)|_{0,\Omega_f} + C|((u_0\cdot\nabla)u_0, u_0)_f|.$$

Applying the estimate for the nonlinear term in Lemma 3.1(1) with $s_1 = 1$, $s_2 = 1/2$ and $s_3 = 0$ and interpolating yields

$$\begin{aligned}|u_t(0)|_{0,\Omega_f} &\le C|\operatorname{div}\varepsilon(u_0)|_{0,\Omega_f} + C|u_0|_{1,\Omega_f}^{3/2}|u_0|_{2,\Omega_f}^{1/2}\\ &\le C|u_0|_{2,\Omega_f} + C|u_0|_{1,\Omega_f}^{3/2}|u_0|_{2,\Omega_f}^{1/2}.\end{aligned}$$

Therefore

$$|u_t(0)|_{0,\Omega_f}^2 \le C[|u_0|_{2,\Omega_f}^2 + |u_0|_{2,\Omega_f}\ |u_0|_{1,\Omega_f}^3].$$

This implies $u_t(0) \in H$. For the wave component, we argue as follows:

$$\begin{aligned}(w_{tt}(t), \psi)_s &= -\Big(\sigma(w(t)), \varepsilon(\psi)\Big)_s + \langle \sigma w(t)\cdot\nu, \psi\rangle\\ &= (\operatorname{div}\sigma(w)(t), \psi)_s.\end{aligned}$$

Next, letting $t \to 0+$ and using assumption $w_0 \in H^2(\Omega_s)$ and the implied $\sigma(w_0)\cdot\nu|_{\Gamma_s} \in H^{1/2}(\Gamma_s)$ we have, for all $\psi \in L_2(\Omega_s)$,

$$\big|(w_{tt}(0), \psi)_s\big| = \big|(\operatorname{div}\sigma(w_0), \psi)_s\big| \le C\big|w_0\big|_{2,\Omega_s}\ \big|\psi\big|_{0,\Omega_s}.$$

Therefore, $w_{tt}(0) \in L_2(\Omega_s)$ as desired. Moreover, as a result of the above,

$$(3.5)\quad |u_t(0)|^2_{0,\Omega_f} + |w_{tt}(0)|^2_{0,\Omega_s} + |w_t(0)|^2_{1,\Omega_s} \le C[|u_0|^4_{1,\Omega_f} + |u_0|^2_{2,\Omega_f} + |w_0|^2_{2,\Omega_s} + |w_1|^2_{1,\Omega_s}]. \qquad \square$$

Lemma 3.4. *If (u, w, w_t) is a weak solution to (1.1), then the following estimate holds provided the right-hand side is finite:*

$$(3.6)\quad |\varepsilon(u)(t)|^2_{0,\Omega_f} \le C[|u_t(t)|^2_{-1,\Omega_f} + |w_{tt}(t)|^2_{0,\Omega_s} + |\varepsilon(w_t)(t)|^2_{0,\Omega_s} + E(0)].$$

Proof. Setting $\varphi = u$ and $\psi = w_t$ in the weak formulation (1.2) and then combining both equations via the continuity of the velocities across Γ_s yields

$$(u_t(t), u(t))_f + |\varepsilon(u(t))|^2_{0,\Omega_f} + (w_{tt}(t), w_t(t))_s + (\varepsilon(w(t)), \varepsilon(w_t(t)))_s + b(u, u, u) = 0.$$

Hence, we can estimate $|\varepsilon(u)|_{0,\Omega_f}$ as follows:

$$\left|\varepsilon(u(t))\right|^2_{0,\Omega_f} \le C\Big[\left|u_t(t)\right|^2_{-1,\Omega_f} + \left|u(t)\right|^2_{0,\Omega_f} + \left|w_{tt}(t)\right|^2_{0,\Omega_s} + \left|w_t(t)\right|^2_{0,\Omega_s} + \left|\sigma(w(t))\right|^2_{0,\Omega_s} + \left|\varepsilon(w_t(t))\right|^2_{0,\Omega_s}\Big].$$

Next, the basic *a priori* estimate—i.e., $u, w_t, w \in L_\infty([0,T]; \mathcal{H})$—obtained in the proof of existence of weak solutions in [3] yields

$$(3.7)\quad |\varepsilon(u)(t)|^2_{0,\Omega_f} \le |u_t(t)|^2_{0,\Omega_f} + |w_{tt}(t)|^2_{0,\Omega_s} + |\varepsilon(w_t)(t)|^2_{0,\Omega_s} + KE(0)$$

for some constant K, thus finishing the proof. $\square$

To conclude this subsection, we note that the condition $Y_t \in L_\infty((0,T); \mathcal{H})$ implies that the right-hand side of of inequality (3.6) is finite. This observation along with the argument used in the proof of Lemma 3.4 implies the following corollary, which improves the regularity stated in Lemma 3.2.

Corollary 3.5. *Let $n = 2, 3$ and let a weak solution satisfy $Y_t \in L_\infty((0,T), \mathcal{H})$. Then $g = \frac{1}{2}(u \cdot \nu)\nu \in L_2((0,T), X')$.*

Proof. From the calculations in Lemma 3.2 we infer

$$\int_0^T \langle g(s), \varphi(s)\rangle\, ds \le C\left(\int_0^T |u(s)|^4_{1,\Omega_f}\, dt\right)^{1/2} |\varphi|_{L_2(0,T,X)},$$

and from Lemma 3.4

$$|g|_{L_2((0,T),X')} \leq C[|Y_t|^2_{L_\infty((0,T),\mathcal{H})} + CE(0)],$$

which implies the desired conclusion. □

3.2. The two-dimensional case.

Theorem 3.6. *Let* Ω *be of dimension* 2*. Then, given*

$$(u_0, w_0, w_1) \in H^2(\Omega_f) \cap V \times H^2(\Omega_s) \times H^1(\Omega_s)$$

satisfying the compatibility conditions (CC) *stated in* (1.7) *and* $T > 0$*, there exists a unique strong solution* (u, w, w_t, p) *satisfying the system* (3.1) *such that*

$$u \in C([0,T]; V) \cap L_2((0,T); H^2(\Omega_f)),$$

$$(w, w_t) \in L_\infty((0,T); H^2(\Omega_s)) \times L_\infty((0,T); H^1(\Omega_s)),$$

$$p \in L_2((0,T); H^1(\Omega_f)).$$

Proof. The proof of Theorem 3.6 follows through the following five steps:
(1) reconstruct additional regularity of time derivatives,
(2) reconstruct the pressure in a strong PDE formulation,
(3) reconstruct regularity of tangential derivatives in the collar neighborhood of the interface Γ_s,
(4) reconstruct tangential regularity of the pressure, and
(5) reconstruct full H^2 regularity of the solutions of u, w along with H^1 regularity of the pressure by appealing to the fact that
  (i) the boundary is non-characteristic, and
  (ii) solution u is divergence free.

The details of this plan are provided below.

Step 1. Regularity of Y_t in $L_\infty([0,T]; \mathcal{H})$.

Lemma 3.7. *Let* $(u_0, w_0, w_1) \in H^2(\Omega_f) \bigcap V \times H^2(\Omega_s) \times H^1(\Omega_s)$*. Assume* (CC) *conditions given in Definition 1.7. Then:*

$$u \in C([0,T]; V), \quad u_t \in L_\infty((0,T); H) \bigcap L_2((0,T); V)$$
$$(w_t, w_{tt}) \in L_\infty([0,T]; H^1(\Omega_s)) \times L_\infty([0,T]; L^2(\Omega_s)). \tag{3.8}$$

Proof. Let $(u_0, w_0, w_1) \in H^2(\Omega_f) \bigcap V \times H^2(\Omega_s) \times H^1(\Omega_s)$ subject to compatibility conditions on the boundary. Then there exists a unique weak solution $(u, w, w_t) \in L_\infty([0,T]; \mathcal{H})$ satisfying (1.2) (cf. Theorem 1.3). Moreover, the derivatives of the initial condition $(u_t(0), w_1, w_{tt}(0))$ are in $\mathcal{H}$ by Lemma 3.3.

For a fixed weak solution $u(t), w(t), w_t(t)$ we consider the following linear problem: Find $\bar{u} \in C_w([0,T],H) \cap L_2((0,T),V)$, $\bar{w} \in C_w([0,T],H^1(\Omega_s))$, $\bar{w}_t \in C_w([0,T],L_2(\Omega_s))$ and such that $\sigma(\bar{w}) \cdot \nu \in L_2((0,T),H^{-1/2}(\Gamma_s))$ and

$$
(3.9) \quad \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}(\bar{u},\varphi)_f + (\varepsilon(\bar{u}),\varepsilon(\varphi))_f + \langle \sigma(\bar{w}) \cdot \nu, \varphi \rangle + b(\bar{u},u,\varphi) + b(u,\bar{u},\varphi) = 0, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}(\bar{w}_t,\psi)_s + (\sigma(\bar{w}),\varepsilon(\psi))_s - \langle \sigma(\bar{w}) \cdot \nu, \psi \rangle = 0, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\bar{w} = \bar{u}, \quad \text{on } \Gamma_s \times (0,T), \\ \bar{u}(0) = u_t(0), \quad \bar{w}(0) = w_t(0), \quad \bar{w}_t(0) = w_{tt}(0) \end{cases}
$$

for all test functions $\varphi \in V$ and $\psi \in H^1(\Omega_s)$. The time derivatives $\mathrm{d}/\mathrm{d}t$ are understood in the distributional sense. The initial conditions at $t = 0$ are well defined in $\mathcal{H}$ on the strength of Lemma 3.3.

The system defined by (3.9) is a linear system to which the theory of weak solutions developed in [3] (see also [5]) applies. In particular the following energy inequality holds:

$$
(3.10) \quad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[|\bar{u}|^2_{0,\Omega_f} + |\bar{w}_t|^2_{0,\Omega_s} + (\sigma(\bar{w}),\varepsilon(\bar{w}))_s \Big] + |\varepsilon(\bar{u}(t))|^2_{0,\Omega_f} + b(\bar{u}(t),u(t),\bar{u}(t)) \le 0.
$$

Let $y(t) = |\bar{u}(t)|^2_{0,\Omega_f} + |\bar{w}_t(t)|^2_{0,\Omega_s} + (\sigma(\bar{w})(t),\varepsilon(\bar{w})(t))$, and use Lemma 3.1 to estimate the trilinear term yielding

$$
\begin{aligned}
(3.11) \quad \frac{\mathrm{d}}{\mathrm{d}t}y(t) + 2|\varepsilon(\bar{u}(t)|^2_{0,\Omega_f} &\le 2|b(\bar{u}(t),u(t),\bar{u}(t))| \\
&\le c_1|\bar{u}(t)|^2_{\frac{1}{2},\Omega_f}|u(t)|_{1,\Omega_f} + c_2|\bar{u}(t)|_{\frac{1}{2},\Omega_f}|u(t)|_{3/4,\Omega_f}|\bar{u}(t)|_{3/4,\Omega_f} \\
&\le c_1|\varepsilon(\bar{u}(t)|_{0,\Omega_f}|\bar{u}(t)|_{0,\Omega_f}|u(t)|_{1,\Omega_f} \\
&\qquad + c_2|\bar{u}(t)|^{3/4}_{0,\Omega_f}|\bar{u}(t)|^{5/4}_{1,\Omega_f}|u(t)|^{3/4}_{1,\Omega_f}|u(t)|^{1/4}_{0,\Omega_f} \\
&\le C_0|\bar{u}(t)|^2_{0,\Omega_f}|u(t)|^2_{1,\Omega_f}(1 + |u(t)|^{2/3}_{0,\Omega_f}) + |\varepsilon(\bar{u}(t)|^2_{0,\Omega_f}.
\end{aligned}
$$

Now integrate (3.11) from 0 to t and absorb the H^1 norm of $\bar{u}$ in the right-hand side

$$
(3.12) \quad y(t) + \int_0^t |\varepsilon(\bar{u})|^2_{0,\Omega_f}\,\mathrm{d}s \le y(0) + C_0\int_0^t |\bar{u}(s)|^2_{0,\Omega_f}\, |u(s)|^2_{1,\Omega_f}(1 + |u(s)|^{2/3}_{0,\Omega_f})\,\mathrm{d}s.
$$

Next we drop the $L_2([0,T];H^1(\Omega_f))$ norm of $\bar{u}$ and apply Gronwall's inequality to

$$y(t) \leq y(0) + C_0 \int_0^t y(s)|u(s)|^2_{1,\Omega_f}(1 + |u(s)|^{2/3}_{0,\Omega_f})\,ds \tag{3.13}$$

yielding

$$y(t) \leq y(0) \exp\left(C_0 \int_0^T |u(s)|^2_{1,\Omega_f}(1 + |u(s)|^{2/3}_{0,\Omega_f})\,ds \right). \tag{3.14}$$

Note that since $u \in L_2([0,T],V)$ with continuous dependence on the initial data $E(0)$, the right-hand side of the inequality (3.14) above is finite. Therefore, using the estimate (3.5) in Lemma 3.3, the inequality (3.14) becomes

$$\begin{aligned} |\bar{u}(t)|^2_{0,\Omega_f} + |\bar{w}_t(t)|^2_{0,\Omega_s} + (\sigma(\bar{w})(t), \varepsilon(\bar{w})(t)) \\ \leq E_1(|u_0|_{2,\Omega_f},\ |w_0|_{2,\Omega_s},\ |w_1|_{1,\Omega_s}) \end{aligned} \tag{3.15}$$

where E_1 is a continuous function in the indicated norms of the initial data. Moreover, if we use (3.15) in (3.12),

$$\int_0^t |\varepsilon(\bar{u})|^2_{0,\Omega_f}\,ds \leq E_2(|u_0|_{2,\Omega_f}, |w_0|_{2,\Omega_s}, |w_1|_{1,\Omega_s}). \tag{3.16}$$

Since u_t, w_t satisfy the same linear equation as $\bar{u}$, $\bar{w}$, we conclude:

$$u_t \in L_\infty(0,T;H) \cap L_2(0,T;V), \tag{3.17}$$

$$(w_t, w_{tt}) \in L_\infty([0,T];H^1(\Omega_s)) \times L_\infty([0,T];L^2(\Omega_s)). \tag{3.18}$$

V regularity of u. In addition, as a consequence of estimate (3.6) in Lemma 3.4 and (3.17), (3.18) we obtain that

$$u \in L_\infty([0,T],V). \tag{3.19}$$

Combining (3.17) with (3.19) implies $u \in H^1([0,T];V)$ and thus, by the Sobolev imbedding,

$$u \in C([0,T];V). \tag{3.20}$$

□

Step 2. Reconstruction of the PDE form (1.1). We apply Lemma 2.4 with $f = (u\cdot\nabla)u$ and $g = -\frac{1}{2}(u\cdot\nu)u$. Indeed, $f \in C([0,T];V')\cap L_\infty(0,T;L_{3/2}(\Omega))$ and $g \in L_2([0,T],X')$ by Lemma 3.1 and Corollary 3.5. Thus, Lemma 2.4 applied with $q = q^* = \frac{3}{2}$ (for $n \le 3$), yields the following PDE form satisfied by (u,w,w_t) and $p \in L_2((0,T);L_2(\Omega_f))$:

$$(3.21)\quad \begin{cases} u_t - \operatorname{div}\mathcal{T}(u,p) + (u\cdot\nabla)u = 0\,, & \Omega_f\times(0,T)\,,\\ \operatorname{div} u = 0\,, & \Omega_f\times(0,T)\,,\\ w_{tt} - \operatorname{div}\sigma(w) = 0\,, & \Omega_s\times(0,T)\,,\\ u(0,\cdot) = u_0\,, & \Omega_f\,,\\ w(0,\cdot) = w_0,\ w_t(0,\cdot) = w_1\,, & \Omega_s\,,\\ w_t = u\,, & \Gamma_s\times(0,T)\,,\\ u = 0\,, & \Gamma_f\times(0,T)\,,\\ \mathcal{T}(u,p)\nu - \sigma(w)\cdot\nu + \frac{1}{2}(u\cdot\nu)u = 0\,, & \text{in } L_2((0,T),H^{-1/2}(\Gamma_s))\,. \end{cases}$$

Step 3. Regularity of the tangential derivatives Our next step is to show that any weak solution driven by sufficiently smooth initial conditions enjoys additional regularity in the space variable. In line with a general strategy of Agmon-Douglis-Nirenberg [1], it suffices to consider the equation in the neighborhood of the boundary (the interior regularity is straightforward and known). In our case, the main issue is the boundary Γ_s where transmission boundary conditions are imposed, which is the most sensitive part of the argument. Thus, in what follows we shall consider equations (3.21) in a small neighborhood of the boundary Γ_s. This is easily accomplished by partition of unity.

We will denote by D_τ a tangential derivative defined in the collar neighborhood of the boundary Γ_s (on both sides of the boundary) . The action of D_τ on vector functions is defined through action on each co-ordinate. The result stated below provides additional regularity of u, w, p, by placing $D_\tau u$, $D_\tau w$, $D_\tau p$ in the appropriate Sobolev spaces.

Lemma 3.8. *Let $n = 2, 3$. Under the assumptions of Theorem 3.6 we have:*

$$(3.22)\quad \begin{aligned} &D_\tau u \in L_2([0,T];H^1(\Omega_f)) \cap C([0,T];L_2(\Omega_f))\,,\\ &D_\tau w \in L_\infty([0,T];H^1(\Omega_s))\,,\\ &D_\tau p \in L_2((0,T);L_2(\Omega_f))\,. \end{aligned}$$

Proof. In order to estimate higher derivatives that are tangential to the boundary Γ_s, we differentiate the whole system (3.21) distributionally in the tangential direction to Γ_s by using the following tangential differential operator S,

$$(3.23)\quad Su = \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i}.$$

S is a first order operator (time independent), with b_i smooth in Ω, such that S is tangent to Γ_s, i.e., $\sum b_i \nu_i = 0$. In order to keep the notation simple, we will not differentiate between S action on scalars and vectors. We apply the tangential differential operator S to the system (3.21). Let $Su = \hat{u}$, $Sw = \hat{w}$, $Sp = \hat{p}$, $S\nu = \hat{\nu}$, and denote by $[D, S]$ the commutator of S with an operator D. S can be thought of as the pre-image under the diffeomorphism via partition of unity from Ω onto the half plane of the tangential derivative ∇_y on the boundary $x = 0$. Here, we are only interested in the resulting local problem in the collar vicinity (on both sides) of the boundary Γ_s.

The system under consideration is the following:

$$\hat{u}_t - \operatorname{div} \mathcal{T}(\hat{u}, \hat{p}) + (\hat{u} \cdot \nabla)u + (u \cdot \nabla)\hat{u} = -[\operatorname{div} \varepsilon, S]u + (u \cdot [\nabla, S])u + [\nabla, S]p\,, \tag{3.24}$$

$$\operatorname{div} \hat{u} = \sum_{j,i=1}^{n} D_j b_i(x) D_i u_j\,, \tag{3.25}$$

$$\hat{w}_{tt} - \operatorname{div} \sigma(\hat{w}) = -[\operatorname{div} \sigma, S]w\,, \tag{3.26}$$

$$\hat{u}(0, \cdot\,) = \hat{u}_0 \in H^1(\Omega_f)\,, \tag{3.27}$$

$$\hat{w}(0, \cdot\,) = \hat{w}_0 \in H^1(\Omega_s), \quad \hat{w}_t(0, \cdot\,) = \hat{w}_1 \in L_2(\Omega_s)\,, \tag{3.28}$$

$$\hat{w}_t|_{\Gamma_s} = \hat{u}|_{\Gamma_s}\,, \tag{3.29}$$

$$\mathcal{T}(\hat{u}, \hat{p}) \cdot \nu - \sigma(\hat{w}) \cdot \nu + \tfrac{1}{2}(u \cdot \nu)\hat{u} + [\sigma \cdot \nu,\ S]w + \tfrac{1}{2}(\hat{u} \cdot \nu)u + \tfrac{1}{2}(u \cdot \hat{\nu})u\Big|_{\Gamma_s} - [\varepsilon \cdot \nu,\ S]u = 0\,. \tag{3.30}$$

The problem (3.24)-(3.30) is linear in the variables $\hat{u}$, $\hat{w}$, $\hat{w}_t$ with *a priori* regularity

$$\begin{aligned} &\hat{u} \in C([0, T], L_2(\Omega_f))\,, \\ &\hat{w} \in C_w([0, T], L_2(\Omega_s)), \quad \hat{w}_t \in C_w([0, T], H^{-1}(\Omega_s))\,, \\ &\hat{p} \in L_2((0, T), H^{-1}(\Omega))\,. \end{aligned} \tag{3.31}$$

Though the system defined above is *linear* in the variables $(\hat{u}, \hat{w})$, it is no longer divergence free in Ω_f. This fact is responsible for the appearance of pressure terms in the equations. Indeed, standard energy method applied to the resulting equation gives the following.

With $\hat{y}$ defined as

$$\hat{y}(t) = |\hat{u}(t)|^2_{0,\Omega_f} + (\sigma(\hat{w}), \varepsilon(\hat{w}))_s + |\hat{w}_t(t)|^2_{0,\Omega_s}, \tag{3.32}$$

we obtain the differential inequality::

$$(3.33)\quad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\hat{y}(t) + |\varepsilon(\hat{u})(t)|^2_{0,\Omega_f} \le -b(\hat{u},u,\hat{u}) - b(u,\hat{u},\hat{u}) + (\hat{p},\operatorname{div}\hat{u})_f - ([\operatorname{div}\varepsilon, S]u,\hat{u})_f + ([\nabla,S]p,\hat{u})_p + ((u\cdot[\nabla,S])u,\hat{u})_f + \frac{1}{2}\langle(u\cdot\hat{\nu})u,\hat{u}\rangle - ([\operatorname{div}\sigma,S]w,\hat{w}_t)_s + \langle[\sigma\cdot\nu,S]w,\hat{w}_t\rangle - \langle[\varepsilon\cdot\nu,S]u,\hat{u}\rangle.$$

It is understood here that the right hand side of the inequality is evaluated at t. We proceed to obtain estimates for the norm of the tangential derivatives $(\hat{u},\hat{w},\hat{w}_t)$ in $L_\infty([0,T];L_2(\Omega_f)\times H^1(\Omega_s)\times L_2(\Omega_s))$ as we did with the time derivatives in Step 1 utilizing the fact that the initial conditions

$$(\hat{u}(0,\cdot),\ \hat{w}(0,\cdot),\ \hat{w}_t(0,\cdot)) \in H^2(\Omega_f)\cap V\times H^1(\Omega_s)\times L^2(\Omega_s).$$

However, the difference is that the new variable $\hat{u}$ is no longer divergence free (unless Γ_s is flat). The estimates carried below are valid for both dimensions $n = 2,3$ since we will use the estimates for the nonlinear term b from Lemma 3.1 that are valid in both dimensions.

We let

$$\begin{aligned}
A_1(t) &= -b(\hat{u},u,\hat{u}) - b(u,\hat{u},\hat{u}) + ((u\cdot[\nabla,S])u,\ \hat{u}) - \tfrac{1}{2}\langle(u\cdot\hat{\nu})u,\hat{u}\rangle\\
A_2(t) &= (\hat{p},\operatorname{div}\hat{u}) + ([\nabla,S]p,\hat{u})\\
A_3(t) &= ([\operatorname{div}\varepsilon,S]u,\hat{u}) - \langle[\varepsilon\cdot\nu,S]u,\hat{u}\rangle\\
A_4(t) &= ([\operatorname{div}\sigma,S]w,\hat{w}_t) + \langle[\sigma\cdot\nu,S]w,\hat{w}_t\rangle.
\end{aligned}$$

I. Estimate for $A_1(t)$

$A_1 = -b(\hat{u},u,\hat{u}) - b(u,\hat{u},\hat{u}) + ((u\cdot[\nabla,S])u,\hat{u}) + \frac{1}{2}\langle(u\cdot\hat{\nu})u,\hat{u}\rangle.$

The first term can be estimated using the usual inequalities for the nonlinear term from Lemma 3.1 with $s_1 = \frac{3}{4}$, $s_2 = 0$ and $s_3 = \frac{3}{4}$. Moreover, the second term is zero by Lemma 3.1 Part 5 since u is divergence free. The third term and the fourth term make another nonlinear term that can be treated in the same way as $b(u,u,\hat{u})$ since $[\nabla,S]$ is a first order differential operator. Hence, by Korn's inequality and interpolation (note $\hat{u}=0$ on Ω_f)

$$\begin{aligned}
(3.34)\quad A_1(t) &\le C|\hat{u}(t)|_{3/4,\Omega_f}|u(t)|_{1,\Omega_f}\,|\hat{u}(t)|_{3/4,\Omega_f} + C|u(t)|^2_{1,\Omega_f}\,|\hat{u}(t)|_{1,\Omega_f}\\
&\le C|\varepsilon(\hat{u})(t)|^{3/2}_{0,\Omega_f}\,|\hat{u}(t)|^{1/2}_{0,\Omega_f}\,|u(t)|_{1,\Omega_f} + |u(t)|^2_{1,\Omega_f}\,|\varepsilon(\hat{u})(t)|_{0,\Omega_f}\\
&\le \delta|\varepsilon(\hat{u})(t)|^2_{0,\Omega_f} + C_1(\delta)[|\hat{u}(t)|^2_{0,\Omega_f}+1]\,|u(t)|^4_{1,\Omega_f}.
\end{aligned}$$

II. Estimate on $A_2(t)$. Recall that

$$A_2(t) = (\hat{p}, \operatorname{div} \hat{u}) + ([\nabla, S]p, \hat{u}). \tag{3.35}$$

Since $p \in L_2(0,T; L_2(\Omega_f))$,

$$\operatorname{div} \hat{u} = \sum_{j,i=1}^{n} D_j b_i(x) D_i u_j$$

(by (3.25)) and $[\nabla, S]$ is a first order space-tangential differential operator, integration by parts in tangential direction yields the following estimate

$$\begin{aligned} A_2(t) &\le C_2 |p(t)|_{0,\Omega_f}\ |D_\tau \operatorname{div}\ \hat{u}|_{0,\Omega_f} + |p|_{0,\Omega_f}\ |D_\tau \hat{u}|_{0,\Omega_f} \\ &\le C |p(t)|_{0,\Omega_f}\ |\hat{u}(t)|_{1,\Omega_f} \\ &\le \delta |\hat{u}(t)|^2_{1,\Omega_f} + C_\delta |p(t)|^2_{0,\Omega_f}. \end{aligned} \tag{3.36}$$

III. Estimate on $A_3(t)$. We begin by noting that a priori estimates for the commutator give rise to the second order differential operators, which, in turn, are too high to be absorbed by the estimates. More refined analysis of the commutators is needed. To accomplish this, by letting ζ_ν and ζ_τ be the corresponding Fourier variables for the normal and tangential coordinates ν and τ, respectively. With this notation, symbolic representation of the principal part of the tangential operator S is the following:

$$Q(\nu, \tau, \zeta_\tau) \sim b(\nu, \tau)\zeta_\tau.$$

We also recall that the principal part of the symbol for the commutator of two linear operators P and Q is a linear combination, denoted by $\{\ \}$, of the following symbols:

$$[P, Q] \sim \left\{ \frac{\partial P}{\partial \zeta_\nu} \frac{\partial Q}{\partial \nu}, \frac{\partial P}{\partial \zeta_\tau} \frac{\partial Q}{\partial \tau}, \frac{\partial P}{\partial \nu} \frac{\partial Q}{\partial \zeta_\tau}, \frac{\partial P}{\partial \tau} \frac{\partial Q}{\partial \zeta_\nu}, \frac{\partial P}{\partial \tau} \frac{\partial Q}{\partial \zeta_\tau} \right\}.$$

This implies that the commutator $[\operatorname{div} \varepsilon, S]$ (written in local coordinates after flattening the boundary [30]) is a second order operator whose principal part is a linear combination of $\{D^2_{\tau,\nu}, D^2_\tau\}$. Moreover, the $[\varepsilon \cdot \nu, S]$ is a first order tangential operator. This is to say

$$[\operatorname{div} \varepsilon, S] \sim \{D^2_{\tau,\nu}, D^2_\tau\} \quad \text{and} \quad [\varepsilon() \cdot \nu, S] \sim \{D_\tau|_{\Gamma_s}\}.$$

We estimate the term A_3 by applying (i) integration by parts in tangential direction, (ii) Sobolev's embeddings and (iii) interpolation inequalities. After accounting for lower order terms in the commutator, we obtain

$$\begin{aligned}
|A_3(t)| &= |([\operatorname{div}\varepsilon, S]u, \hat{u})_f - \langle[\varepsilon\cdot\nu, S]u, \hat{u}\rangle| \\
&\le C(|D_\nu u(t)|_{0,\Omega_f}|D_\tau\hat{u}(t)|_{0,\Omega_f} + (|D_\tau u(t)|_{0,\Omega_f}|D_\tau\hat{u}(t)|_{0,\Omega_f} \\
&\qquad + |D_\tau u(t)|_{-1/2,\Gamma_s}|\hat{u}(t)|_{1/2,\Gamma_s} + |u(t)|^2_{1,\Omega_f}) \\
&\le \delta|(\hat{u})(t)|^2_{1,\Omega_f} + C_\delta|u|^2_{1,\Omega_f} \\
&\le \delta|\varepsilon(\hat{u})(t)|^2_{0,\Omega_f} + C_\delta|u(t)|^2_{1,\Omega_f}.
\end{aligned}$$

IV. Estimate on $A_4(t)$.

$$A_4(t) = ([\operatorname{div}\sigma, S]w, \hat{w}_t) + \langle[\sigma\cdot\nu, S]w, \hat{w}_t\rangle.$$

As in III, the principal part of the commutator $[\operatorname{div}\sigma, S]w$ is a second order differential operator of the form $\operatorname{span}\{D^2_{\tau,\nu}, D^2_\tau\}$, while the principal part of $[\sigma\cdot\nu, S]$ is a first order tangential differential operator of the form D_τ. Thus we estimate $\int_0^T A_4(t)\,\mathrm{d}t$ as follows.

$$\left|\int_0^T A_4(t)\,\mathrm{d}t\right| = \left|\int_0^T ([\operatorname{div}\sigma, S]w, \hat{w}_t)_s + \langle[\sigma\cdot\nu, S]w, \hat{w}_t\rangle\,\mathrm{d}t\right|.$$

Integration in time-space tangential direction yields

$$\begin{aligned}
&\le C\int_0^T |D_\nu, w_t|_{0,\Omega_s}|D_\tau\hat{w}|_{0,\Omega_s} + |D_\tau w_t|_{0,\Omega_s}|D_\tau\hat{w}|_{0,\Omega_s} \\
&\qquad + C\Big[[|D_\nu w|_{0,\Omega_s} + |D_\tau w|_{0,\Omega_s}]\ |D_\tau\hat{w}|_{0,\Omega_s}\Big]\Big|_0^T \\
&\qquad + C\int_0^T |w_t|_{1/2,\Gamma_s}\ |D_\tau\hat{w}|_{-1/2,\Gamma_s} + C\sup_{t\in[0,T]}|D_\tau w(t)|_{-1/2,\Gamma_s}\ |\hat{w}(t)|_{1/2,\Gamma_s}.
\end{aligned}$$

By the standard trace theorem,

$$\le C\sup_{t\in[0,T]}|w(t)|_{1,\Omega_s}\ |\hat{w}(t)|_{1,\Omega_s} + C\int_0^T |w_t|_{1,\Omega_s}\ |\hat{w}|_{1,\Omega_s} \quad \le \tag{3.37}$$

$$\leq \delta \sup_{t\in[0,T]} |\hat{w}(t)|^2_{1,\Omega_s}$$

$$+ C_\delta \sup_{t\in[0,T]} |w(t)|^2_{1,\Omega_s} + \delta \int_0^T |\hat{w}|^2_{1,\Omega_s}\, \mathrm{d}t + C_\delta \int_0^T |w_t|^2_{1,\Omega_s}\, \mathrm{d}t.$$

Now, collecting estimates I-IV (3.34)-(3.37) for the terms in (3.33) and rewriting estimates on an interval $(0,t)$ yields:

$$\hat{y}(t) + \int_0^t |\varepsilon(\hat{u})(t)|^2_{0,\Omega_f} \tag{3.38}$$

$$\leq \hat{y}(0) + C_\delta \int_0^t [|\hat{u}|^2_{0,\Omega_f} + 1]\, |u(t)|^4_{1,\Omega_f} + |p(t)|^2_{0,\Omega_f} + |u(t)|^2_{1,\Omega_f} + |w_t(t)|^2_{1,\Omega_s}\, \mathrm{d}z$$

$$+ C_\delta \sup_{z\in(0,t)} |w(z)|^2_{1,\Omega_s} + \delta \sup_{z\in[0,t]} |\hat{w}(t)|^2_{1,\Omega_s} + \delta \int_0^t |\hat{u}(t)|_{1,\Omega_f}\, \mathrm{d}z\,.$$

Taking sup with respect to $t \in [0,T]$ allows us to absorb the two terms with δ, where the latter is selected suitably small. The remaining terms on the right hand side with δ are defined and finite since we have shown in Step 1 that

$$u \in L_\infty([0,T];V)$$

and thus

$$u \in L_\infty([0,T];H^1(\Omega_f))\,,$$

while $p \in L_2([0,T];L_2(\Omega_f))$ and $w \in W^1_\infty([0,T];L_2(\Omega_s))$. Letting

$$M \equiv \sup_{t\in[0,T]} \Big[[|u(t)|^2_{1,\Omega_f} + 1]\, |u(t)|^4_{1,\Omega_f} + |u(t)|^2_{1,\Omega_f} + |w_t|^2_{1,\Omega_s} \Big] + \int_0^T |p|^2_{0,\Omega_f}\, \mathrm{d}t\,, \tag{3.39}$$

we obtain for an arbitrary $t \in [0,T]$

$$\hat{y}(t) + \int_0^t |\varepsilon(\hat{u})(s)|^2_{0,\Omega_f}\, \mathrm{d}s \leq \hat{y}(0) + TM\,. \tag{3.40}$$

Hence,

$$\hat{y}(t) \leq \hat{y}(0) + TM(|u_0|_{2,\Omega_f},\ |w_0|_{2,\Omega_s},\ |w_1|_{1,\Omega_s}) \tag{3.41}$$

and

$$\int_0^t |\varepsilon(\hat{u})(s,\cdot)|^2_{0,\Omega_f}\,\mathrm{d}s \le E(|u_0|_{2,\Omega_f}, |w_0|_{2,\Omega_s}, |w_1|_{1,\Omega_s}, T)\,, \tag{3.42}$$

concluding the desired tangential regularity of (u,w) stated in the first two inequalities in Lemma 3.8.

Step 4: Tangential regularity of the pressure. The argument applies to $n = 2$ and $n = 3$. By using the improved tangential regularity of u, w obtained in Step 3, and noting that $S\nabla p = Su_t - S div\, \varepsilon(u) + S(u\cdot\nabla)u$ we obtain (see (3.48) below for the L_2 regularity of nonlinear term) that $S\nabla p \in L_2((0,T); H^{-1}(\Omega_f))$. Since $S\nabla p = \nabla Sp + [\nabla, S]p$, $p \in L_2((0,T); L_2(\Omega_f))$, and $[\nabla, S]p \in L_2((0,T); H^{-1}(\Omega_f)$, the above implies $\nabla\hat{p} \in L_2((0,T); H^{-1}(\Omega_f)$. Since $\hat{p} \in L_2((0,T); H^{-1}(\Omega_f))$,

$$\hat{p} \in L_2(0,T; L_2(\Omega_f)). \tag{3.43}$$

The proof of tangential regularity asserted in Lemma 3.8 is thus completed. □

Step 5: H^2 regularity of w.

Lemma 3.9. *Under the assumptions of Theorem 1.8*

$$w \in C_w([0,T], H^2(\Omega_s)), \tag{3.44}$$

$$\sigma(w)\cdot\nu \in L_\infty(0,T; H^{1/2}(\Gamma_s)). \tag{3.45}$$

Proof. Lemma 3.8 implies $\hat{w}|_{\Gamma_s} \in L_\infty([0,T]; H^{1/2}(\Gamma_s))$ and therefore

$$w|_{\Gamma_s} \in L_\infty([0,T]; H^{3/2}(\Gamma_s)). \tag{3.46}$$

Now, $\hat{u} \in L_2([0,T]; H^1(\Omega_f))$ and hence $u|_{\Gamma_s} \in L_2([0,T]; H^{3/2}(\Gamma_s))$. We can use this information to boost the regularity of the elastic equation since we also have $w_{tt} \in L_\infty([0,T]; L_2(\Omega_s))$ (by Lemma 3.7 in Step 1). This leads to the following elliptic Dirichlet problem:

$$\mathrm{div}\,\sigma(w) = w_{tt} \text{ in } L_\infty([0,T]; L_2(\Omega_s))$$

$$w|_{\Gamma_s} \in C_w([0,T]; H^{3/2}(\Gamma_s)).$$

Standard elliptic theory completes the argument. □

Step 6 : H^2 regularity of u and H^1 regularity of p

Lemma 3.10. *Under the assumption of Theorem 3.6 above we obtain that* $u \in L_2((0,T);H^2(\Omega_f))$ *and* $p \in L_2((0,T);H^1(\Omega_f))$.

Proof. *L_2 space-time regularity of the nonlinear term $(u\cdot\nabla)u$ in 2 and 3 dimensions.*

We present the 3d estimate. Denote by $u \in X_1(X_2(X_3))$ a mixed X_1-regularity in time, X_2-regularity in the normal direction and X_3-regularity in the tangential plane, and by $u \in X_1(X_2)$ a mixed X_1-regularity in time and X_2-regularity in space.

From the previous estimates, $u \in L_2(L_2(H^2))$ and $u \in L^2(H^1(H^1))$. Interpolation implies $u \in L_2(H^\theta(H^{2-\theta}))$ for all θ in $[0,1]$. This (for any θ, $\frac{1}{2} < \theta < 1$) in turn yields

$$u \in L_2(C_0). \tag{3.47}$$

by the Sobolev imbedding.

On the other hand, the previous estimates also give $u \in C_0(H^1)$—hence, $\nabla u \in C_0(L_2)$. This together with (3.47) implies the desired $L_2(L_2)$-regularity of the nonlinear term

$$(u\cdot\nabla)u \in L_2((0,T)\times\Omega)) \tag{3.48}$$

H^2 space regularity of u. We recall standard notation $D_\nu \equiv \nabla\cdot\nu$ and $D_\tau \equiv \nabla\cdot\tau$, where τ denotes unit tangential vector defined in a collar neighborhood of the boundary. Denote $z \equiv D_\nu u$, $v \equiv D_\tau u$.

Let us start by collecting the regularity already available.

$$\begin{aligned} v = D_\tau u \in L_2((0,T);H^1(\Omega_f)), \quad & D_\nu v \in L_2((0,T);L_2(\Omega_f)), \\ D_\tau z \in L_2((0,T);L_2(\Omega_f)), \quad & \text{div } z \in L_2((0,T);L_2(\Omega_f)) \end{aligned} \tag{3.49}$$

where the regularity of div z follows from the fact that $div\ u = 0$ and *a priori* regularity of the variable u.

Moreover, the tangential regularity of the pressure (3.43) along with already established (time-tangential) regularity of u and the original PDE equation satisfied by u imply $div\ \varepsilon(u)\cdot\tau \in L_2((0,T);L_2(\Omega_f))$, which translates into

$$D_\nu z\cdot\tau = D_\nu^2 u\cdot\tau \in L_2((0,T);L_2(\Omega_f)). \tag{3.50}$$

Thus, in order to establish H^2 regularity of u it suffices to prove the membership of $D_\nu z\cdot\nu$ in L_2. For this, we shall use the last statement in (3.49) along with

(3.50). Indeed, expressing the divergence operator in terms of D_ν and D_τ as

$$\operatorname{div} z = D_\nu(n_1 z_1) - D_\tau(n_2 z_1) + D_\nu(n_2 z_2) + D_\tau(n_1 z_2), \tag{3.51}$$

$$z = (z_1, z_2), \quad \nu = (n_1, n_2), \tag{3.52}$$

and using tangential regularity of z, $D_\tau z \in L_2((0,T); L_2(\Omega_f))$ we obtain

$$D_\nu(n_1 z_1) + D_\nu(n_2 z_2) \in L_2((0,T); L_2(\Omega_f)).$$

The above combined with commutator rules and the already established regularity of z yield

$$n_1 D_\nu z_1 + n_2 D_\nu z_2 \in L_2((0,T); L_2(\Omega_f)). \tag{3.53}$$

On the other hand, explicitly rewriting (3.50) gives

$$-n_2 D_\nu z_1 + n_1 D_\nu z_2 \in L_2((0,T); L_2(\Omega_f)). \tag{3.54}$$

Since the determinant of the matrix

$$\begin{pmatrix} -n_2 & n_1 \\ n_1 & n_2 \end{pmatrix}$$

is equal to -1, we obtain $D_\nu z \in L_2((0,T); L_2(\Omega))$. Hence, from (3.49),

$$u \in L_2((0,T); H^2(\Omega_f)), \tag{3.55}$$

and from the equation, $D_\nu p \in L_2((0,T); L_2(\Omega_f))$, which implies via (3.43) that

$$p \in L_2((0,T); H^1(\Omega_f)) \tag{3.56}$$

—as stated in the theorem. □

The proof of Theorem 1.8 is thus completed. □

3.3. Strong solutions in three dimensions

3.3.1. Local-in-time solutions for general data

Theorem 3.11. *Let Ω be of dimension 3. Then given*

$$(u_0, w_0, w_1) \in H^2(\Omega_f) \cap V \times H^2(\Omega_s) \times H^1(\Omega_s)$$

satisfying the compatibility conditions (CC) *stated in* (1.7)*, there exists $T' > 0$ and a unique strong solution (u, w, w_t, p) satisfying the system* (3.1) *such that*

$$\begin{aligned} &u \in C([0,T']; V) \cap L_2((0,T'); H^2(\Omega_f)), \\ &(w, w_t) \in L_\infty((0,T'); H^2(\Omega_s)) \times L_\infty((0,T'); H^1(\Omega_s)), \\ &p \in L_2((0,T'); H^1(\Omega_f)). \end{aligned}$$

Proof. Regularity of time derivatives.

We again have the existence of a weak solution to (1.2)

$$(u, w, w_t) \in C_w([0,T]; \mathcal{H}),$$

and that the time derivatives at the initial time $(u_t(0), w_1, w_{tt}(0)) \in \mathcal{H}$ by Lemma 3.3. As before, we estimate the norms of the time derivatives in $L_\infty([0,T]; \mathcal{H})$ (as in (3.11)). To accomplish this we fix weak solution solution u, w and consider the same linear problem as defined in (3.9) for the variables $\bar{u}, \bar{w}$. The following energy estimate holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[|\bar{u}|^2_{0,\Omega_f} + |\bar{w}_t|^2_{0,\Omega_s} + (\sigma(\bar{w}), \varepsilon(\bar{w}))_{0,\Omega_s}\Big] + 2\big|\varepsilon(\bar{u})(t)\big|^2_{0,\Omega_f} \leq \Big|b(\bar{u}(t),\ u(t),\ \bar{u}(t))\Big|. \tag{3.57}$$

Let $y(t) = |\bar{u}(t)|^2_{0,\Omega_f} + |\bar{w}_t(t)|^2_{0,\Omega_s} + (\sigma(\bar{w}(t), \varepsilon(\bar{w}(t))_{0,\Omega_s} + KE(0) + 1$ and use the estimate for the trilinear term in 3d from Lemma 3.1 to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}y(t) + 2|\varepsilon(\bar{u}(t)|^2_{0,\Omega_f} &\leq C|\varepsilon(\bar{u}(t)|^{3/2}_{0,\Omega_f}|\bar{u}(t)|^{1/2}_{0,\Omega_f}|u(t)|_{1,\Omega_f} \\ &\leq |\varepsilon(\bar{u}(t)|^2_{0,\Omega_f} + C|\bar{u}(t)|^2_{0,\Omega_f}|u(t)|^4_{1,\Omega_f}.\end{aligned} \tag{3.58}$$

Similarly as before, we absorb the H^1 norm of u_t from the right hand side into the left hand side of (3.58) and then drop the term to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) \leq C|\bar{u}(t)|^2_{0,\Omega_f}|\varepsilon(u)(t)|^4_{0,\Omega_f}. \tag{3.59}$$

On the other hand, (see 3.6) we infer that

$$|\varepsilon(u)(t)|^2_{0,\Omega_f} \leq C[|\bar{u}(t)|^2_{0,\Omega_f} + |\bar{w}_t(t)|^2_{0,\Omega_s} + |\varepsilon(\bar{w}(t)|^2_{0,\Omega_s} + E(0)]. \tag{3.60}$$

Hence

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}y(t) &\leq C_0|\bar{u}(t)|^2_{0,\Omega_f} \\ &\quad\times \big[|\bar{u}(t)|^2_{0,\Omega_f} + |\bar{w}_t(t)|^2_{0,\Omega_s} + |\varepsilon(\bar{w})(t)|^2_{0,\Omega_s} + KE(0) + 1\big]^2\end{aligned} \tag{3.61}$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}t}y \leq C_0 y^3, \tag{3.62}$$

which gives

$$y(t) \leq \sqrt{\frac{1}{(1/y^2(0)) - 2C_0 t}},$$

i.e.,

$$(3.63)\quad |u_t(t)|^2_{0,\Omega_f} + |w_{tt}(t)|^2_{0,\Omega_s} + (\sigma(w_t), \varepsilon(w_t))_{0,\Omega_s} \leq \sqrt{\frac{1}{(1/y^2(0)) - 2C_0 t}},$$

where $y(0) = |u_t(0)|^2_{0,\Omega_f} + |w_{tt}(0)|^2_{0,\Omega_s} + (\sigma(w_1), \varepsilon(w_1))_{0,\Omega_s} + E(0) + 1$ which is finite. Hence, $(u_t, w_t, w_{tt}) \in L_\infty([0, T']; \mathcal{H})$ where $T' = 1/(2C_0 y^2(0))$. Moreover, from (3.6) we see that $u \in L_\infty([0, T']; V)$, and if we integrate (3.58) in time, we also conclude that $u_t \in L_2([0, T']; V)$.

The estimates/regularity results presented in Steps 2-6 are already valid in 3d. The only difference is that the second part of Step 6 requires an analogous 3d-decomposition of the divergence operator in the normal and the tangential components. The details are provided below.

Let $\nu = (n_1, n_2, n_3)$ be the unit normal vector to the boundary Γ_s while τ and κ are two linearly independent tangential unit vectors to the boundary Γ_s at a given point $p \in \Gamma_s$. Let q be any point in Ω_f such that q has the same x and y coordinates as p. With ν, τ and κ at q defined to be those at p, we rewrite $z = D_\nu u$ at any such q as $z = (z\cdot\nu)\nu + (z\cdot\tau)\tau + (z\cdot\kappa)\kappa$. Thus,

$$\begin{aligned}
\operatorname{div} z &= \operatorname{div}(z\cdot\nu)\nu + \operatorname{div}(z\cdot\tau)\tau + \operatorname{div}(z\cdot\kappa)\kappa \\
&= \partial_1[(z_1n_1+z_2n_2+z_3n_3)n_1 + (z_1\tau_1+z_2\tau_2+z_3\tau_3)\tau_1 + (z_1\kappa_1+z_2\kappa_2+z_3\kappa_3)\kappa_1] \\
&\quad + \partial_2[(z_1n_1+z_2n_2+z_3n_3)n_2 + (z_1\tau_1+z_2\tau_2+z_3\tau_3)\tau_2 + (z_1\kappa_1+z_2\kappa_2+z_3\kappa_3)\kappa_2] \\
&\quad + \partial_3[(z_1n_1+z_2n_2+z_3n_3)n_3 + (z_1\tau_1+z_2\tau_2+z_3\tau_3)\tau_3 + (z_1\kappa_1+z_2\kappa_2+z_3\kappa_3)\kappa_3] \\
&= \nabla(z_1n_1)\cdot\nu + \nabla(z_2n_2)\cdot\nu + \nabla(z_3n_3)\cdot\nu + \nabla(z_1\tau_1)\cdot\tau + \nabla(z_2\tau_2)\cdot\tau \\
&\qquad + \nabla(z_3n_3)\cdot\tau + \nabla(z_1\kappa_1)\cdot\kappa + \nabla(z_2\kappa_2)\cdot\kappa + \nabla(z_3\kappa_3)\cdot\kappa \\
&\qquad + (z\cdot\nu)\operatorname{div}\nu + (z\cdot\tau)\operatorname{div}\tau + (z\cdot\kappa)\operatorname{div}\kappa \\
&= D_\nu(z_1n_1) + D_\nu(z_2n_2) + D_\nu(z_3n_3) + D_\tau(z_1\tau_1) + D_\tau(z_2\tau_2) \\
&\qquad + D_\tau(z_3n_3) + D_\kappa(z_1\kappa_1) + D_\kappa(z_2\kappa_2) + D_\kappa(z_3\kappa_3) \\
&\qquad + (z\cdot\nu)\operatorname{div}\nu + (z\cdot\tau)\operatorname{div}\tau + (z\cdot\kappa)\operatorname{div}\kappa.
\end{aligned}$$

We now choose

$$\tau = \frac{1}{\sqrt{n_3^2+n_1^2}}(n_3, 0, -n_1) \quad \text{and} \quad \kappa = \frac{1}{\sqrt{n_3^2+n_1^2}}(-n_1n_2, n_3^2+n_1^2, -n_3n_2).$$

Therefore,

$$\begin{aligned}\operatorname{div} z &= D_\nu(z_1 n_1) + D_\nu(z_2 n_2) + D_\nu(z_3 n_3)\\ &\quad + D_\tau\left(z_1 \frac{1}{\sqrt{n_3^2+n_1^2}} n_3\right) - D_\tau\left(z_3 \frac{1}{\sqrt{n_3^2+n_1^2}} n_1\right)\\ &\quad - D_\kappa\left(z_1 \frac{1}{\sqrt{n_3^2+n_1^2}} n_1 n_2\right) + D_\kappa(z_2\sqrt{n_3^2+n_1^2})\\ &\quad - D_\kappa\left(z_3 \frac{1}{\sqrt{n_3^2+n_1^2}} n_3 n_2\right) + (z\cdot\nu)\operatorname{div}\nu + (z\cdot\tau)\operatorname{div}\tau + (z\cdot\kappa)\operatorname{div}\kappa.\end{aligned}$$

Now all the tangential derivatives $D_\tau z, D_\kappa z \in L_2([0,T];L_2(\Omega_f))$ and τ,κ are C^1 functions while $\operatorname{div} z \in L_2([0,T];L_2(\Omega_f))$. Thus,

$$D_\nu(z_1 n_1) + D_\nu(z_2 n_2) + D_\nu(z_3 n_3) \in L_2([0,T];L_2(\Omega_f)).$$

With $\nu \in C^1$ and $z \in L_2([0,T];L_2(\Omega_f))$ we then have

$$n_1 D_\nu z_1 + n_2 D_\nu z_2 + n_3 D_\nu z_3 \in L_2([0,T];L_2(\Omega_f)). \tag{3.64}$$

On the other hand, since we know the tangential derivative of u is in $L_2([0,T];H^1(\Omega_f))$, we also have $D_\nu(z\cdot\tau) = D_{\nu\nu}(u\cdot\tau) \in L_2([0,T];L_2(\Omega_f))$ and similarly $D_\nu(z\cdot\kappa) \in L_2([0,T];L_2(\Omega_f))$. Thus,

$$\frac{n_3}{\sqrt{n_3^2+n_1^2}} D_\nu z_1 - \frac{n_1}{\sqrt{n_3^2+n_1^2}} D_\nu z_3 \in L_2([0,T];L_2(\Omega_f)), \tag{3.65}$$

$$\frac{-n_1 n_2}{\sqrt{n_3^2+n_1^2}} D_\nu z_1 + \sqrt{n_3^2+n_1^2} D_\nu z_2 - \frac{n_3 n_2}{\sqrt{n_3^2+n_1^2}} D_\nu z_3 \in L_2([0,T];L_2(\Omega_f)). \tag{3.66}$$

The equations (3.64) - (3.66) produce a system

$$A D_\nu z = b \in [L_2([0,T];L_2(\Omega_f))]^3,$$

where

$$A = \begin{pmatrix} n_1 & n_2 & n_3\\ \frac{n_3}{\sqrt{n_3^2+n_1^2}} & 0 & -\frac{n_1}{\sqrt{n_3^2+n_1^2}}\\ -\frac{n_1 n2}{\sqrt{n_3^2+n_1^2}} & \sqrt{n_3^2+n_1^2} & -\frac{n_3 n_2}{\sqrt{n_3^2+n_1^2}} \end{pmatrix}.$$

The coefficients of the matrix are continuous, and the determinant of this matrix is 1. This gives the desired result $D_\nu z \in L_2([0,T];L_2(\Omega_f))$, hence $u \in L_2((0,T);H^2(\Omega))$. From the equation we then obtain $p \in L_2((0,T0,H^1(\Omega)$, as desired. ◻

3.3.2. Global-in-time solutions for small initial data

Theorem 3.12. *Let Ω be of dimension 3, and let $(u_0, w_0, w_1) \in H^2(\Omega_f) \cap V \times H^2(\Omega_s) \times H^1(\Omega_s)$ such that*

$$|u_0|^6_{1,\Omega_f} + |u_0|^4_{2,\Omega_f} + |w_0|^2_{2,\Omega_s} + |w_1|^2_{1,\Omega_s} \le C$$

for a suitable absolute constant C. Then, there exists a unique strong solution (u, w, w_t, p) satisfying the system (3.1) such that

$$\begin{aligned}
&u \in C([0,\infty);V) \cap L_2((0,\infty);H^2(\Omega_f)),\\
&(w, w_t) \in L_\infty((0,\infty);H^2(\Omega_s)) \times L_\infty((0,\infty);H^1(\Omega_s)),\\
&p \in L_2((0,\infty);H^1(\Omega_f)).
\end{aligned}$$

Proof. We proceed as in proof of Theorem 3.11 to estimate the norms of the derivatives (u_t, w_t, w_{tt}) of a weak solution in $\mathcal{H}$. Starting from estimate (3.57) and letting $y(t) = |\bar{u}(t)|^2_{0,\Omega_f} + |\bar{w}_t(t)|^2_{0,\Omega_s} + (\sigma(\bar{w}(t)), \varepsilon(\bar{w}(t)))_{0,\Omega_s}$, we estimate the term $b(\bar{u}, u, \bar{u})$ via Lemma 3.1 with $s_1 = s_3 = 1, s_2 = 0$ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) + 2|\varepsilon(\bar{u})(t)|^2_{0,\Omega_f} \le C_1 |\varepsilon(\bar{u})(t)|^2_{0,\Omega_f} |\varepsilon(u)(t)|_{0,\Omega_f} \tag{3.67}$$

and consequently

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) + (2 - C_1|\varepsilon(u)(t)|_{0,\Omega_f})|\varepsilon(\bar{u})(t)|^2_{0,\Omega_f} \le 0. \tag{3.68}$$

On the other hand, we have from (3.6) and (3.5) that

$$\begin{aligned}
|\varepsilon(u_0)|^2_{0,\Omega_f} &\le |u_t(0)|^2_{0,\Omega_f} + |w_{tt}(0)|^2_{0,\Omega_s} + |\varepsilon(w_t)|^2_{0,\Omega_s} + KE(0)\\
&\le y(0) + KE(0)\\
&\le (C+K)[|u_0|^6_{1,\Omega_f} + |u_0|^2_{2,\Omega_f} + |w_0|^2_{2,\Omega_s} + |w_1|^2_{1,\Omega_s}]\\
&< \frac{4}{C_1^2}
\end{aligned}$$

where the last inequality follows from the assumption on the initial condition. Therefore $(2 - C_1|\varepsilon(u_0)|_{0,\Omega_f}) \ge 0$. Since we already have a local in time result

$u \in C([0,T'];V)$ (see Theorem 3.11), we can assume that there exists the first time instant of time T_0 such that $(2 - C_1|\varepsilon(u)(T_0)|_{0,\Omega_f}) = 0$.

Hence, for $t \leq T_0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) \leq 0 \tag{3.69}$$

and thus

$$y(t) \leq y(0) < \frac{4}{C_1^2}. \tag{3.70}$$

Now, from (3.6), for $t \leq T_0$,

$$|\varepsilon(u)(t)|^2_{0,\Omega_f} \leq y(t) + KE(0) \leq y(0) + KE(0) < \frac{4}{C_1^2}.$$

Thus, $(2 - C_1|\varepsilon(u)(t)(t)|_{0,\Omega_f}) > 0$ for $t \leq T_0$ which contradicts assumption that $(2 - C_1|\varepsilon(u)(T_0)|_{0,\Omega_f}) = 0$. Therefore, $y(t) \leq 4/(C_1^2)$ for all $t > 0$.

This proves that $(u_t, w_t, w_{tt}) \in L_\infty((0,\infty);\mathcal{H})$. Steps 2-6 proceed exactly as in the local-in-time case but on time interval $(0,\infty)$ recovering analogous results globally in time.

□

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VIOREL BARBU:
Romanian Academy, Iasi, Romania
E-MAIL: barbu@acad.ro

IRENA LASIECKA:
Department of Mathematics
University of Virginia
Charlottesville, VA 22904, U.S.A.
E-MAIL: il2v@virginia.edu

AMJAD TUFFAHA:
Department of Mathematics
University of Southern California
Los Angeles, CA 90089, U.S.A
E-MAIL: tuffaha@usc.edu

ZORAN GRUJIĆ:
Department of Mathematics
University of Virginia
Charlottesville, VA 22904, U.S.A.
E-MAIL: zg7c@virginia.edu

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