Interpolation Between Algebraic and Geometric Conditions for Smoothness of the Vorticity in the 3D NSE

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ABSTRACT. A sufficient condition for smoothness of the vorticity in the 3D NSE containing a space-time integrability condition on the vorticity magnitude and a Hölder-like condition on the vorticity direction as the 'end-point' cases is given.

1. INTRODUCTION

The well-posedness of the 3D NSE remains fundamentally unresolved. One direction that is being pursued is formulation of various conditions that would guarantee smoothness of the solutions on some time interval $(0, T)$. A well-known set of conditions is a mixed space-time $L^q - L^r$ boundness of the velocity field in a certain range of q's and r's,

$$||u||_q^{2q/(q-3)} \in L^1(0,T) \quad \text{for some } q > 3,$$

including the limit case $q = \infty$, and the localized versions [7, 11, 12]. A related condition expressed in terms of smallness of the weak-L^3 space norm uniformly in time can be found in [9], and a related BMO-condition in [10].

Analogous regularity conditions on the vorticity field ω, $\omega = \operatorname{curl} u$, follow easily from the space-time integrabilty conditions on ∇u,

$$||\nabla u||_q^{2q/(2q-3)} \in L^1(0,T) \quad \text{for some } q,\ 3 \le q \le \infty;$$

see [2], and the Calderon-Zygmund theorem. One obtains that

$$||\omega||_q^{2q/(2q-3)} \in L^1(0,T) \quad \text{for some } q,\ 3 \le q < \infty$$

suffices to retain the smoothness (the case $q = \infty$ is included via a different argument). Studying the vorticity field is interesting since the vorticity is a more geometric quantity than the velocity is. Constantin [5] discovered that a streching factor in the evolution of the vorticity magnitude can be depleted by the local alignment of the vorticity directions. This was subsequently utilized in [6] to show that a Lipschitz-like regularity of the vorticity direction suffices to control the growth of the vorticity magnitude and thus prevent a blow-up. More recently, Beirao da Veiga and Berselli [3] proved an analogous theorem assuming $\frac{1}{2}$-Hölder-like regularity of the vorticity direction.

The goal of this paper is to derive a more general sufficient condition for smoothness of the vorticity that would contain the limit ($q = \infty$) space-time integrability of the vorticity magnitude, an 'algebraic' condition, and the condition on the Hölder-like regularity of the vorticity direction, a 'geometric' condition, as the special 'end-point' cases.

2. Algebraic Conditions

Consider the vorticity formulation of the 3D NSE,

$$\omega_t - \nu\Delta\omega + (u\cdot\nabla)\omega = (\omega\cdot\nabla)u, \tag{2.1}$$

where ω is the vorticity of the fluid, u is the velocity which can be recovered by the Biot-Savart law

$$u(x) = -\frac{1}{4\pi}\int \nabla\frac{1}{|y|}\times\omega(x+y)\,\mathrm{d}y,$$

and $\nu > 0$ is the viscosity (the spatial domain Ω will be $\mathbb{R}^3$). The right-hand-side term in (2.1) is the genuine 3D term which is responsible for the streching of the vorticity magnitude, and is not present in the 2D-case.

Beirao da Veiga [2] formulated a sufficient condition for the regularity in terms of the space-time integrability of the velocity gradient,

$$||\nabla u||_q^{2q/(2q-3)} \in L^1(0,T) \quad \text{for some } q,\ 3\le q\le\infty. \tag{2.2}$$

Since $\omega = \operatorname{curl} u$, one can write $\nabla u = (\nabla\circ\operatorname{curl}^{-1})\omega$, i.e., ∇u is expressed as a Calderon-Zygmund operator of ω. That, combined with (2.2), yields the following set of algebraic conditions for smoothness of the vorticity:

$$||\omega||_q^{2q/(2q-3)} \in L^1(0,T) \quad \text{for some } q,\ 3\le q<\infty. \tag{2.3}$$

The limit case

$$\|\omega\|_\infty \in L^1(0,T) \tag{2.4}$$

also guarantees the regularity (this was originally derived in [1] for the 3D Euler, and it holds for the 3D NSE as well). More recently, Kozono and Taniuchi [10] showed that this condition can be weakened to

$$\|\omega\|_{\mathrm{BMO}} \in L^1(0, T).$$

The main ingredient in their proof are some precise bilinear estimates in BMO (the standard space of bounded mean oscillations).

3. Geometric Conditions

We start with fixing the notation. In the vortical region $\Omega^* = \{x \in \Omega : |\omega(x)| > 0\}$ the vorticity direction ξ is well-defined, $\xi = \omega/|\omega|$. Recall that the strain tensor (or the deformation tensor) S is defined by $S = \frac{1}{2}(\nabla u + \nabla u^T)$. A key quantity will be α, which is given as $\alpha = S\xi \cdot \xi$.

Constantin [5] discovered that α effectively controls the evolution of the vorticity magnitude

$$(\partial_t + u \cdot \nabla - \nu\Delta)|\omega|^2 + \nu|\nabla\omega|^2 = S\omega \cdot \omega = \alpha|\omega|^2, \tag{3.1}$$

and that α can be *geometrically* depleted. Namely, the following integral representation for α holds:

$$\alpha(x) = \frac{3}{4\pi} P.V. \int D\left(\frac{y}{|y|}, \xi(x+y), \xi(x)\right) |\omega(x+y)| \frac{dy}{|y|^3}, \tag{3.2}$$

where the geometric factor D is proportional to the volume spanned by the unit vectors $y/|y|$, $\xi(x+y)$, and $\xi(x)$. More precisely,

$$D(e_1, e_2, e_3) = (e_1 \cdot e_3)\,\mathrm{Det}(e_1, e_2, e_3) \tag{3.3}$$

for any triplet of unit vectors e_1, e_2, and e_3. Denoting the angle between $\xi(x+y)$ and $\xi(x)$ by φ, it is easy to see that $|D| \le |\sin\varphi| \le |\xi(x+y) - \xi(x)|$, and so a certain regularity of ξ, i.e., a local alignment of the vorticity directions, will deplete the singularity in (3.2). This is very significant since both numerical simulations and the experiments revealed that the regions of high vorticity magnitude tend to organize into the structures with strong local alignment (e.g., vortex sheets and vortex tubes).

The above considerations were exploited in [6], where it was proved that as long as the vorticity direction satisfies a Lipschitz-like regularity, no blow-up can occur. For $M > 0$, $t > 0$, denote by $\Omega_t(M)$ the super-level sets

$$\Omega_t(M) = \{x \in \Omega : |\omega(x,t)| \ge M\}.$$

Theorem 3.1 (Constantin-Fefferman). *Assume that there exist constants c, $M > 0$ such that*

$$|\sin\varphi| \le c|y|$$

for all $x \in \Omega_t(M)$, $t \in (0,T)$. Then $\lim_{t\uparrow T} \|\omega(t)\|_2 < \infty$.

Following [6] approach, Beirao da Veiga and Berselli [3] proved that 1/2-Hölder-like regularity of ξ suffices.

Theorem 3.2 (Beirao da Veiga-Berselli). *Assume that there exist constants c, $M > 0$ such that*

$$|\sin\varphi| \le c|y|^{1/2},$$

for all $x \in \Omega_t(M)$, $t \in (0,T)$. Then $\lim_{t\uparrow T} \|\omega(t)\|_2 < \infty$.

Remark 3.3. The appearance of a Hölder condition had been conjectured by A.R. in 1998 based on a different argument.

Remark 3.4. Another geometric condition on the vorticity guaranteeing the smoothness was given in [8]. It is essentially a condition on the local existence of a sparse/thin direction in the regions of high vorticity magnitude on the length scales comparable to a localized ω-version of the Kolmogorov scale. The proof relies on L^∞-type estimates on the complexified solutions and a plurisubharmonic measure maximum principle in $\mathbb{C}^3$.

4. CONTROL OF $\|\omega\|_q$ VIA THE GEOMETRIC DEPLETION OF α

The goal of this section is to formulate a sufficient condition for smoothness of the vorticity that would naturally include an algebraic condition (2.4) and the geometric condition in Theorem 3.2 as the end-point cases.

Theorem 4.1. *Let $u_0 \in L^2$, $\operatorname{div} u_0 = 0$, $\omega_0 = \operatorname{curl} u_0 \in L^1 \cap L^q$ for some $q \ge 2$, and assume that there exist absolute constants c and M such that a solution ω satisfies*

(i) $\|\omega\|_q^{q/(q-1)} \in L^1(0,T)$,

(ii) $|\sin\varphi(\xi(x+y),\xi(x))| \le c|y|^{1/q}$ *for all $x \in \Omega_t(M)$, $t \in (0,T)$.*

Then $\lim_{t\uparrow T} \|\omega(t)\|_q < \infty$.

Remark 4.2. The case $q = 2$ recovers Theorem 3.2, and the limit case $q = \infty$ recovers the space-time limit condition (2.4).

Remark 4.3. We present formal *a priori* estimates. A rigorous proof is easily obtained by the standard continuation argument.

Remark 4.4. The proofs of Theorems 3.1 and 3.2 are based on controlling the L^2-norm of ω (the enstrophy) via the geometric depletion of α. Our proof is based on controlling the L^q-norms ($2 \le q \le \infty$) of ω via the geometric depletion of α.

Proof. We start by rewriting (2.1) as

$$\omega_t - \nu\Delta\omega + (u \cdot \nabla)\omega = S\omega. \tag{4.1}$$

Taking the dot product with $|\omega|^{2p-2}\omega$ ($p \geq 1$), integrating by parts, and utilizing that u is divergence-free, we arrive at

$$\begin{aligned}\frac{1}{2p}\frac{d}{dt}\int |\omega|^{2p}\,\mathrm{d}x + \nu\int |\omega|^{2p-2}\sum_j |\partial_j\omega|^2\,\mathrm{d}x \\ + 2\nu(p-1)\int |\omega|^{2p-4}\sum_j(\partial_j\omega\cdot\omega)^2\,\mathrm{d}x = \int |\omega|^{2p-2}S\omega\cdot\omega,\end{aligned} \tag{4.2}$$

where the unadorned integrals are the integrals over $\mathbb{R}^3$. The first ν-term can be written as

$$\frac{\nu}{p^2}\sum_k\int |\nabla(\omega_k^p)|^2\,\mathrm{d}x,$$

and we simply drop the second ν-term. That yields

$$\frac{d}{dt}\int |\omega|^{2p}\,\mathrm{d}x + \frac{2\nu}{p}\sum_k\int |\nabla(\omega_k^p)|^2\,\mathrm{d}x \leq 2p\int |\omega|^{2p-2}S\omega\cdot\omega. \tag{4.3}$$

Since our geometric condition is assumed only in the regions of high vorticity magnitude, we decompose the vorticity field accordingly. Write

$$\omega(t) = \omega^{(1)}(t) + \omega^{(2)}(t),$$

where

$$\omega^{(1)}(t) = (1 - \Xi_{\Omega_t(M)})\omega(t) \quad \text{and} \quad \omega^{(2)}(t) = \Xi_{\Omega_t(M)}\omega(t).$$

(Ξ denotes the characteristic function of a set.) This induces a decomposition of the strain tensor S,

$$S = S^{(1)} + S^{(2)},$$

where

$$S^{(i)} = \frac{1}{2}((\nabla\circ\mathrm{curl}^{-1})\omega^{(i)} + ((\nabla\circ\mathrm{curl}^{-1})\omega^{(i)})^T)$$

for $i = 1, 2$. To estimate the nonlinear term

$$NL = \int |\omega|^{2p-2}S\omega\cdot\omega$$

we consider 3 different possibilities:

(a) There exists an $\omega^{(1)}$-factor.

$$|NL| \le M \int |S|\,|\omega|^{2p-1}\,\mathrm{d}x \le M\|S\|_{2p}\,\||\omega|^{2p-1}\|_{2p/(2p-1)} \le cpM \int |\omega|^{2p}\,\mathrm{d}x, \tag{4.4}$$

where the Calderon-Zygmund theorem was used in the last step.

(b) $S^{(i)} = S^{(1)}$.

$$|NL| \le \int |S^{(1)}|\,|\omega|^{2p}\,\mathrm{d}x \le c_p \sum_k \int |S^{(1)}|\omega_k^{2p}\,\mathrm{d}x \le c_p\|S^{(1)}\|_m \sum_k \|\omega_k^{2p}\|_n, \tag{4.5}$$

where m and n are conjugate exponents. Utilizing the Calderon-Zygmund theorem and the fact that $\|\omega\|_1$ is bounded in terms of the initial data [4], (4.5) leads to

$$|NL| \le c_p c_0^{1/m} m M^{(m-1)/m} \sum_k \|\omega_k^{2p}\|_n, \tag{4.6}$$

where c_0 depends on the initial data. Rewriting $\|\omega_k^{2p}\|_n$ as $\|\omega_k^p\|_{2n}^2$ and applying the Gagliardo-Nirenberg interpolation inequality for $2 < 2n \le 6$ we obtain

$$|NL| \le c_{0,p,\nu} m^{2n/(3-n)} M^{2/(3-n)} \int |\omega|^{2p}\,\mathrm{d}x + \frac{\nu}{2p^2} \sum_k \int |\nabla(\omega_k^p)|^2\,\mathrm{d}x. \tag{4.7}$$

Note that n and m can be chosen such that the prefactor of $\|\omega\|_{2p}^{2p}$ is up to a logarithmic correction linear in M (for M large). More precisely, requiring that $M^{2/(3-n)} = M\log M$ leads to

$$n = \frac{1+3\varepsilon}{1+\varepsilon} \text{ and } m = \frac{1+3\varepsilon}{2\varepsilon}, \quad \text{where } \varepsilon = \frac{\log\log M}{\log M}.$$

Assuming that $\varepsilon \le \frac{1}{3}$ (i.e., M large enough) yields

$$m^{2n/(3-n)} M^{2/(3-n)} \le \frac{1}{\varepsilon^2} M \log M \le (\log M)^3 M,$$

leading to the final bound

$$|NL| \le c_{0,p,\nu} (\log M)^3 M \int |\omega|^{2p}\,\mathrm{d}x + \frac{\nu}{2p^2} \sum_k \int |\nabla(\omega_k^p)|^2\,\mathrm{d}x. \tag{4.8}$$

(c) All $\omega^{(i)}$ are $\omega^{(2)}$ and $S^{(i)} = S^{(2)}$. (Henceforth we drop the superscripts.) This is the case in which the assumption on the regularity of ξ holds—however, for clarity of the exposition, we will at this point assume

$$|\sin\varphi| \le c|y|^{\varepsilon}, \tag{4.9}$$

and choose $0 < \varepsilon = \varepsilon(p) < 3$ later.

Hölder inequality followed by the Hardy-Littlewood-Sobolev inequality yield

$$\begin{aligned} |NL| &\le \int |\alpha|\,|\omega|^{2p}\,\mathrm{d}x \le \||\omega|^{2p}\|_{r_2/(r_2-1)}\,\|\alpha\|_{r_2} \\ &\le c\||\omega|^{2p}\|_{r_2/(r_2-1)}\,\|\omega\|_{3r_2/(3+\varepsilon r_2)} \\ &\le c_p\Big(\sum_k \|\omega_k^p\|^2_{2r_2/(r_2-1)}\Big)\Big(\sum_k \|\omega_k^p\|^{1/p}_{3r_2/(p(3+\varepsilon r_2))}\Big), \end{aligned} \tag{4.10}$$

for any $3/(3-\varepsilon) < r_2 < \infty$. It turns out that applying the Gagliardo-Nirenberg interpolation inequality gives the bound independent of r_2 as long as $r_2 > r_2^*(p,\varepsilon)$, and that the optimal choice for ε is $\varepsilon = 1/(2p)$. Setting e.g. $r_2 = 3p$, the nonlinearity is bounded by

$$c_p\Big(\sum_k \|\omega_k^p\|^2_{6p/(3p-1)}\Big)\Big(\sum_k \|\omega_k^p\|_2^{1/p}\Big) \le c_p\||\omega|^p\|_2^2\Big(\sum_k \|\nabla(\omega_k^p)\|_2^{1/p}\Big). \tag{4.11}$$

Hence,

$$|NL| \le c_{p,\nu}\|\omega\|_{2p}^{2p/(2p-1)}\int |\omega|^{2p}\,\mathrm{d}x + \frac{\nu}{2p^2}\sum_k \int |\nabla(\omega_k^p)|^2\,\mathrm{d}x. \tag{4.12}$$

Collecting the estimates (a), (b), and (c), and setting $q = 2p$ gives the desired differential inequality

$$\frac{d}{dt}\int |\omega|^q\,\mathrm{d}x \le c_{0,q,\nu}\big((\log M)^3 M + \|\omega\|_q^{q/(q-1)}\big)\int |\omega|^q\,\mathrm{d}x. \tag{4.13}$$

□

Remark 4.5. (i) M can be time-dependent; in particular, we can take $M \sim \|\omega\|_\infty^{1/2}$ (up to a logarithmic correction).

(ii) The Hölder regularity condition on ξ needs to hold only locally. On the other hand, one can *derive* Hölder-like regularity on *very* small scales via the known estimates on $\nabla\omega$.

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REFERENCES

[1] J.T. BEALE, T. KATO, and A. MAJDA, *Remarks on the breakdown of smooth solutions for the 3-D Euler equations*, Comm. Math. Phys. **94** (1984), 61–66. MR 85j:35154

[2] H. BEIRÃO DA VEIGA, *A new regularity class for the Navier-Stokes equations in* $\mathbf{R}^n$, Chinese Ann. Math. Ser. B **16** (1995), 407–412. MR 96m:35035

[3] HUGO BEIRÃO DA VEIGA and LUIGI C. BERSELLI, *On the regularizing effect of the vorticity direction in incompressible viscous flows*, Differential Integral Equations **15** (2002), 345–356. MR 2002k:35248

[4] PETER CONSTANTIN, *Navier-Stokes equations and area of interfaces*, Comm. Math. Phys. **129** (1990), 241–266. MR 91i:35148

[5] ______, *Geometric statistics in turbulence*, SIAM Rev. **36** (1994), 73–98. MR 95d:76057

[6] PETER CONSTANTIN and CHARLES FEFFERMAN, *Direction of vorticity and the problem of global regularity for the Navier-Stokes equations*, Indiana Univ. Math. J. **42** (1993), 775–789. MR 95j:35169

[7] E.B. FABES, B.F. JONES, and N.M. RIVIÈRE, *The initial value problem for the Navier-Stokes equations with data in* L^p, Arch. Rational Mech. Anal. **45** (1972), 222–240. MR 47 #5463

[8] ZORAN GRUJIĆ, *The geometric structure of the super-level sets and regularity for 3D Navier-Stokes equations*, Indiana Univ. Math. J. **50** (2001), 1309–1317. MR 2002i:35146

[9] HIDEO KOZONO, *Removable singularities of weak solutions to the Navier-Stokes equations*, Comm. Partial Differential Equations **23** (1998), 949–966. MR 99g:35103

[10] HIDEO KOZONO and YASUSHI TANIUCHI, *Bilinear estimates in BMO and the Navier-Stokes equations*, Math. Z. **235** (2000), DOI 10.1007/s002090000130, 173–194, http://dx.doi.org/10.1007/s002090000130. MR 2001g:76011

[11] GIOVANNI PRODI, *Un teorema di unicità per le equazioni di Navier-Stokes*, Ann. Mat. Pura Appl. (4) **48** (1959), 173–182. MR 23 #A3384 (Italian)

[12] JAMES SERRIN, *On the interior regularity of weak solutions of the Navier-Stokes equations*, Arch. Rational Mech. Anal. **9** (1962), 187–195. MR 25 #346

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