The Geometric Structure of the Super-level Sets and Regularity for 3D Navier-Stokes Equations

ZORAN GRUJIĆ

ABSTRACT. We utilize L^∞-estimates on the complexified solutions of 3D Navier-Stokes equations (NSE) via plurisubharmonic measure to show that, under certain geometric assumptions on the structure of the super-level sets, blow-up in a finite time can not occur.

1. INTRODUCTION

Local existence and uniqueness of regular solutions of 3D NSE are well-known. As an illustration, if $\nabla u_0 \in L^2(\Omega)$, there exists $T^*(\|\nabla u_0\|_{L^2}) > 0$ such that u is regular on $(0, T^*)$. Moreover, for as long as it is regular, u is in analytic Gevrey class [9].

However, except for small initial data, the problem of global regularity for 3D NSE is still open. There are various sufficient conditions for the regularity, e.g., boundedness of solutions in $L^{p,q}$ space-time spaces, $q = 2p/(p-3)$ (see, e.g., [18] for $p > 3$, and [16] and [7] for the borderline case $p = 3$). An alternative approach to the problem is to obtain an estimate on the size of a (possible) singular set. The best result in that direction is obtained in [5], and it implies that for every $T > 0$, one-dimensional Hausdorff measure of the singular set in $\Omega \times (0, T)$ is 0 (see also [8]).

It has been observed both in numerical simulations and experiments that for large values of Reynolds number Re, solutions of 3D Navier-Stokes equations tend to concentrate on sparse geometric structures – the regions on which the magnitude of a solution is above a certain threshold (the regions of "active fluid") are sparse. In particular, if one considers the vorticity formulation, the regions of intense vorticity appear to be dominated by thin, quasi low-dimensional structures. Some examples of such structures are quasi one-dimensional "vortex tubes" and quasi two-dimensional "vortex sheets". (For numerical evidence, see e.g., [1], [14], [20], and [21].) A possible mechanism behind the formation of thin sets

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has been investigated in [10], where the alignment between the vorticity and the intermediate eigenvector of the strain matrix has been studied in the context of both inviscid and viscous 3D flows.

In this paper, we precisely formulate a notion of "being sparse" for the super-level sets $\Omega_t(M) = \{x \in \Omega : |u(x,t)| \geq M\}$ (M large) that will enable us to effectively utilize the estimates on the complexified solutions via plurisubharmonic measure, and show that finite time blow-up in this scenario can not occur. This result is in the spirit similar to [4] in which an assumption on the local alignment of vorticity directions (also a geometric assumption) is shown to deplete the nonlinearity preventing finite time blow-up.

The proof reveals that in order to prevent the blow-up, the regions of active fluid need to be sparse only on a length scale comparable to the radius of spatial analyticity – hence the radius of spatial analyticity emerges as a minimal geometrically significant length scale in the apparent sparse organization of the flow.

We would like to stress that our assumptions are purely geometric, we do not assume anything on the measure of $\Omega_t(M)$.

The paper is organized as follows. In Section 2, we recall some estimates on the complexified solutions from [13], while in Section 3 we state some basic properties of plurisubharmonic measure. Section 4 contains the main result.

2. NOTATION AND SOME KNOWN RESULTS

We consider the NSE in $\Omega = [0,L]^3$ with potential forces, and periodic boundary conditions.

$$\begin{aligned} &\frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla\pi = 0, \\ &\nabla\cdot u = 0 \end{aligned} \tag{2.1}$$

for $(x,t) \in \Omega \times (0,\infty)$, supplemented with the initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega,$$

and

$$\int_\Omega u(x,t)\,dx = 0, \quad t \geq 0, \tag{2.2}$$

where $\mathbb{R}^3$-valued function u is the velocity, $\mathbb{R}$-valued function π is the pressure, $\nu > 0$ is the viscosity, and $\nabla\cdot u_0 = 0$.

For the properties of solutions the reader is referred to [3].

Let $R_0 = L/\nu\|u_0\|_{L^\infty}$ denote the Reynolds number of the initial configuration. Then, the following theorem holds.

Theorem 2.1 ([13]). *Let $T = L^2/[c\nu R_0^2(1+\log_+ R_0)^2]$. Then the solution u of* (2.1), (2.2) *on* $(0,T]$ *satisfies the following property: for every* $t \in (0,T]$, u *is* *the restriction of a complex analytic function* $u + iv$ *with the domain of analyticity containing the region* $\mathcal{R}_t$,

$$\mathcal{R}_t = \{x + iy \in \mathbb{C}^3 : |y| \le c^{-1}(\nu t)^{1/2}\}.$$

Moreover,

$$\|u(\cdot, y, t)\|_{L^\infty} + \|v(\cdot, y, t)\|_{L^\infty} \le c\|u_0\|_{L^\infty}$$

for all $t \in (0, T]$ *and* $(x, y) \in \mathcal{R}_t$.

Remark 2.2. u is unique in the class of all weak solutions of (2.1), (2.2).

Henceforth, we will use the following notation. Let S be a set in $\mathbb{C}^n$. Then, for a complex valued function f defined on S, we set

$$\|f\|_{H^\infty(S)} = \sup_{z \in S} |f(z)|.$$

Also, c, c_i denote positive universal constants that may differ from line to line.

Remark 2.3. Notice that there exists an *absolute* constant K such that the uniform analyticity radius of u attained at $t = T$ is at least

$$\rho(T) = \frac{1}{K}\frac{L}{R_0(1 + \log_+ R_0)}, \tag{2.3}$$

and

$$\|u(T)\|_{H^\infty(\mathcal{R}_T)} \le K\|u_0\|_{L^\infty}. \tag{2.4}$$

3. Plurisubharmonic measure

We briefly recall the definition and some basic properties of the plurisubharmonic measure, as well as a generalization of the Two-Constants Theorem that will be used in the following section.

Let $D \subset \mathbb{C}^n$ be an open set, and $E \subset \partial D$. Define $P[E] = \{u \in \mathrm{PSH}(D) : u \le 1, \limsup_{z \to x \in E} u(z) \le 0\}$, where $\mathrm{PSH}(D)$ denotes the set of all functions plurisubharmonic on D (one can think about plurisubharmonic functions as being subharmonic on every complex line). The function $\omega_{\complement E} = \sup_{P[E]} u$ is called the plurisubharmonic measure of $\complement E$.

The plurisubharmonic measure extends to higher complex dimensions the one-dimensional concept of harmonic measure. Whenever $\omega_{\complement E}$ is of class C^2, it satisfies the complex Monge-Ampère equation

$$\det\left(\left(\frac{\partial^2}{\partial z^j \partial \overline{z^k}}\right)(\omega_{\complement E})\right) = 0$$

which is in a sense a generalization of the Laplace equation that is compatible with the complex structure of $\mathbb{C}^n$.

One occurrence in which plurisubharmonic measure clearly generalizes 1D harmonic measure is the following version of the Two-Constants Theorem for $\mathbb{C}^n$.

Theorem 3.1 ([11]). *Let $D \subset \mathbb{C}^n$ be an open set, and $E \subset \partial D$. Assume that f is a bounded analytic function on D satisfying the following property:*

$$\|f\|_{H^\infty(E)} \leq M \quad \text{and} \quad \|f\|_{H^\infty(\complement E)} \leq N$$

(*the inequalities are in the sense of* lim sup *through D*). *Then*

$$|f(z)| \leq M^{1-\omega_{\complement E}(z)} N^{\omega_{\complement E}(z)}$$

for all $z \in D$.

Unfortunately, it is very hard to explicitly compute the plurisubharmonic measure, except in the cases where the sets in question have a lot of symmetries; for some explicit computations see [2]. However, if both D and E have a product structure, the computation of the plurisubharmonic measure can be reduced to the computation of one-dimensional harmonic measures [12]. In particular, the following form of Theorem 3.1. holds.

Proposition 3.2. *Let $D_1, D_2, \ldots, D_n$ be open sets in $\mathbb{C}$, and let $E_1, E_2, \ldots, E_n$ be such that $E_i \subset \partial D_i$, for $i = 1, 2, \ldots, n$. Denote by ω_{E_i} the harmonic measures of E_i with respect to D_i, and assume that f is a bounded analytic function on $D = D_1 \times D_2 \times \cdots \times D_n$ satisfying the following property:*

$$\|f\|_{H^\infty(E_i)} \leq M_2, \quad \|f\|_{H^\infty(\complement E_i)} \leq M_1$$

for $i = 1, 2, \ldots, n$ (the inequalities are in the sense of lim sup *through D_i). Then*

$$|f(z)| \leq M_2^{\inf(\omega_{E_1}(z), \omega_{E_2}(z), \ldots, \omega_{E_n}(z))} M_1^{1-\inf(\omega_{E_1}(z), \omega_{E_2}(z), \ldots, \omega_{E_n}(z))}$$

for all $z \in D$.

Notice that Proposition 3.2 implies that the sparseness of the set $E = E_1 \times E_2 \times \cdots \times E_n$ in one direction suffices to improve the estimate on f.

The following result shows that we can analytically distort E (in a non-degenerate manner), and still get the same estimate.

Proposition 3.3 ([19]). *Let $F : D \to \mathbb{C}^n$ be an analytic map from a domain $D \subset \mathbb{C}^n$ such that* rank $F = n$ *at some point $a \in D$. If a domain D^* contains $F(D)$, then for any $E^* \subset D^*$, $\omega_{\complement E^*}(F(z)) \leq \omega_{\complement F^{-1}E^*}(z)$ for all $z \in D$.*

4. Sparse super-level sets

We precisely formulate a notion of "being sparse" for the super-level sets $\Omega_t(M) = \{x \in \Omega : |u(x,t)| \geq M\}$, and show that, in this scenario, blow-up in a finite time can not occur.

Definition 4.1. For $u_0 \in L^\infty$, set $M_0 = (1/K^3)\|u_0\|_{L^\infty}$ (K is the absolute constant in (2.3), (2.4)), and consider a solution u of (2.1), (2.2) with the initial datum u_0. Let $\tau > 0$. We say that the set $\Omega_\tau(M_0)$ is *sparse* if for every cube I^3 with side-length

$$\lambda^1(I) = \frac{1}{K_1 K}\frac{L}{R_0(1+\log_+ R_0)}$$

there exists a coordinate projection Π^i satisfying

$$\text{(A)} \qquad \frac{\lambda^1(\Pi^i(\Omega_\tau(M_0) \cap I^3))}{\lambda^1(I)} \leq K_2$$

where $K_1 = \tan(\pi/8) = \sqrt{2} - 1$ and $K_2 = (\tan(\pi/8))^2 = (\sqrt{2}-1)^2$. Here, λ^1 is the one-dimensional Lebesgue measure.

Remark 4.2. The assumption (A) is a requirement that, for large enough levels M, the super-level sets are sparse on a length scale comparable to $L/[R_0(1+\log_+ R_0)]$ – essentially the inverse scale of the Reynolds number of the initial configuration. Since that is at the same time a lower bound on the radius of spatial analyticity, the following theorem indicates that the analyticity radius is in a sense a minimal geometrically significant length scale.

Theorem 4.3. *Let $u_0 \in L^\infty$, and let u be a solution of* (2.1), (2.2) *with the initial datum u_0. Assume that the super-level sets $\Omega_\tau(M_0)$ are sparse for all τ for which $\|u(\tau)\|_{L^\infty} > \|u_0\|_{L^\infty}$. Then u is regular for all times, and $\|u(t)\|_{L^\infty} \leq K\|u_0\|_{L^\infty}$ for all $t \geq 0$.*

Remark 4.4. u is unique in the class of all weak solutions of (2.1), (2.2).

Proof. The idea of the proof is to show that we can iteratively (in time) solve the NSE starting from u_0 infinitely many times – each time for an interval of regularity greater or equal to

$$T = \frac{L^2}{c\nu R_0^2(1+\log_+ R_0)^2}. \tag{4.1}$$

It is enough to explain the first iteration – it will be clear that we can apply the same reasoning for the whole sequence.

Solve the NSE with $u(0) = u_0$. Then, by Section 2,

$$\|u(T)\|_{H^\infty(\{|y|\leq\rho(T)\})} \leq K\|u_0\|_{L^\infty} \tag{4.2}$$

where

$$\rho(T) = \frac{1}{K}\frac{L}{R_0(1+\log_+ R_0)}. \tag{4.3}$$

Consider three cases:

(i) $\|u(T)\|_{L^\infty} < \|u_0\|_{L^\infty}$,
(ii) $\|u(T)\|_{L^\infty} = \|u_0\|_{L^\infty}$,
(iii) $\|u(T)\|_{L^\infty} > \|u_0\|_{L^\infty}$.

For (i), either $\|u(t)\|_{L^\infty} < \|u_0\|_{L^\infty}$ for all $t \geq T$, which gives us regularity for all times, or $\|u(t)\|_{L^\infty} = \|u_0\|_{L^\infty}$ for some $t > T$, and we can do the second iteration starting from the same L^∞-level of the initial data.

For (ii), just do the second iteration.

(iii) is the only interesting case. Since $\|u(T)\|_{L^\infty} > \|u_0\|_{L^\infty}$, the super level set $\Omega_T(M_0)$ is sparse, and our goal is to show that sparseness will actually prevent this case, and thus only (i) or (ii) can happen.

Fix $w \in \{z = x + iy \in \mathbb{C}^3 : y_k = \rho(T)/2,\ k = 1,2,3\}$, and let I^3 be the cube in the real space closest to w,

$$\lambda^1(I) = \frac{1}{K_1 K}\frac{L}{R_0(1+\log_+ R_0)}.$$

Then, there exists a projection $\Pi^{i(w)}$ such that

$$\frac{\lambda^1(\Pi^{i(w)}(\Omega_T(M_0)\cap I^3))}{\lambda^1(I)} \leq K_2. \tag{4.4}$$

Consider now the $i(w)$th projection of the positive part of the domain of analyticity of u at time T, i.e., a 2π-periodic strip

$$D_T^{i(w)} = \{z = x + iy \in \mathbb{C} : 0 \leq y \leq \rho(T)\},$$

and let $w_{i(w)} = x_{i(w)} + iy_{i(w)} \in D_T^{i(w)}$ be the $i(w)$th component of w. Then, since the harmonic measure of any part of the boundary of $D_T^{i(w)}$ with respect to $D_T^{i(w)}$ computed at $w_{i(w)}$ is up to a normalization $(\times[1/(2\pi)])$ just the angle at which the point $w_{i(w)}$ sees that part of the boundary, a simple geometric consideration yields that for $K_1 = \tan(\pi/8)$

$$\omega_{(\{y=\rho(T)\}\cup\{|x-x_{i(w)}|\geq[1/(2K_1K)][L/(R_0(1+\log_+ R_0))]\})}(w_{i(w)}) \leq \frac{5}{8}. \tag{4.5}$$

Similarly, if

$$\frac{\lambda^1\left(\left\{|x - x_{i(w)}| \le \frac{1}{2K_1K}\frac{L}{R_0(1+\log_+ R_0)}\right\} \cap \Omega_T(M_0)\right)}{\frac{1}{K_1K}\frac{L}{R_0(1+\log_+ R_0)}} \le K_2 \tag{4.6}$$

where $K_2 = (\tan(\pi/8))^2$, then

$$\omega_{(\{|x-x_{i(w)}|\le[1/(K_1K)][L/(R_0(1+\log_+ R_0))]\}\cap\Omega_T(M_0))}(w_{i(w)}) \le \frac{1}{8}. \tag{4.7}$$

The inequality (4.4) implies the needed conditions; hence, combining (4.5) and (4.7),

$$\omega_{(\{y=\rho(T)\}\cup\{\Pi^{i(w)}(\Omega_T(M_0))_{\text{per}}\})}(w_{i(w)}) \le \frac{3}{4}. \tag{4.8}$$

Let D_T denote the positive part of the domain of analyticity of u at time T,

$$D_T = \{z = x + iy \in \mathbb{C}^3 : 0 \le y_k \le \rho(T),\ k = 1,2,3\}.$$

Then

$$\|u(T)\|_{H^\infty(\partial D_T)} \le K^4 M_0 \tag{4.9}$$

and

$$|u(T,z)| \le M_0 \tag{4.10}$$

for $z \in \partial D_T \setminus \{\{z = x+iy \in \mathbb{C}^3 : y_k = \rho(T),\ k = 1,2,3\} \cup (\Omega_T(M_0))_{\text{per}}\}$. Since the harmonic measure estimate (4.8) is uniform in w, Proposition 3.2 together with the estimates (4.9) and (4.10) imply that

$$|u(T,w)| \le (K^4M_0)^{3/4}{M_0}^{1/4} = K^3M_0 = \|u_0\|_{L^\infty} \tag{4.11}$$

for all $w \in \{z = x + iy \in \mathbb{C}^3 : y_k = \rho(T)/2,\ k = 1,2,3)\}$.

By symmetry (the complexified solution u satisfies $u(\bar{z}) = \overline{u(z)}$), the same estimate holds for the negative part of domain as well, and thus

$$\|u(T)\|_{H^\infty(\{z=x+iy\in\mathbb{C}^3:|y_k|=\rho(T)/2,\ k=1,2,3\})} \le \|u_0\|_{L^\infty}. \tag{4.12}$$

Finally, the maximum principle yields

$$\|u(T)\|_{L^\infty} \le \|u_0\|_{L^\infty} \tag{4.13}$$

contradicting (iii). ☐

Remark 4.5. While it appears that locally we need a sparse coordinate direction, that condition can be relaxed. Namely, since the plurisubharmonic measure does not increase under suitable regular distortions of a set (see Proposition 3.3), more general sparse geometry can be handled by the same techniques. In particular, the local existence of *any* sparse direction suffices.

Remark 4.6. The weakest known (uniform on a time interval) weak-L^p condition needed for the regularity is "small ε" boundedness locally in weak-L^3 [15]. One can readily check that (A) implies no weak-L^p local boundedness, and so it is a genuine geometric condition. Moreover, the aforementioned weak-L^3 result can be derived utilizing the techniques presented here as the geometrically worst case scenario.

Addendum. A sharp version of local analytic smoothing of the vorticity equations in L^∞ has been recently established in [17]. Utilizing an analog of Theorem 2.1 for the vorticity $\omega = \nabla \times u$, we obtain an analog of Theorem 4.3 for ω; the length scale of sparseness needed to prevent the blow-up of $\|\omega(t)\|_{L^\infty}$ is

$$\frac{\nu^{1/2}}{\|\omega_0\|_{L^\infty}^{1/2}(1+\log_+(L^2\|\omega_0\|_{L^\infty}/\nu))^{1/2}}.$$

A simple way of realizing locally sparse directions is the presence of locally thin geometric structures, e.g., vortex tubes. There is numerical evidence (see [14]), and also some rigorous estimates [6], that (suitably defined) width of a vortex tube scales with (a suitable version of) the Kolmogorov length. It is worth pointing out that the length scale of sparseness needed to control the L^∞-norm is, in the regions of high vorticity magnitude, bounded from *below* (up to a logarithmic correction) by a localized Kolmogorov length, and so there emerges from our theory a possible scenario explaining how the sparse geometry of the flow (e.g., vortex tubes) depletes the nonlinear effects in 3D NSE.

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Department of Mathematics
University of Virginia
Kerchof Hall
Charlottesville, VA 22904, U. S. A.
E-MAIL: `zg7c@virginia.edu`

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