Boundary Control Model of a Nonlinear System of Fluid-Structure Interactions

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***Abstract*— We consider a boundary control model for a coupled system of a Navier Stokes equation coupled with a dynamic system of elasticity that models fluid structure interaction in two and three dimensions. We first establish existence of weak and strong solutions for the uncontrolled model, and then proceed to establish the admissibility of a certain class of boundary controls**

I. INTRODUCTION

We consider a model of fluid-structure interaction defined on a bounded domain $\Omega \in R^n$, $n = 2, 3$, where Ω is comprised of two open domains Ω_f and Ω_s. A stationary solid Ω_s is fully immersed in a fluid on a domain Ω_f with interaction taking place on the boundary of the solid Γ_s. The dynamics of the solid are described by a linear wave equation in the variable w, while the dynamics of the fluid are described as usual by a non-stationary Navier Stokes equation in the variable u. The interaction between the two systems takes place on the boundary Γ_s that is common to both media and is prescribed via suitable (Neumann type) transmission boundary conditions. The model presented is well established in both physical and mathematical literature [16], [9], [7], [10], [8], [18]. From the physical point of view, it is an important model arising in a variety of applications in cell biology, mechanics and fluid dynamics. From the mathematical point of view, the interest in the model stems from the rather unusual functional analytic setup of the model that is not amenable to the standard variational analysis usually employed to study Navier Stokes equations or wave equations. This is due to boundary conditions which are of Neumann type and, as such, can not be treated via standard Leray projectors. The presence of the pressure term, both in the fluid equation and on the boundary is a characteristic feature of the problem. This, in turn, leads to a weak formulation involving the trace of the wave component that is not classically defined in the topology of weak solutions.

In order to cope with the issue, the results of the previous literature have considered an "approximation" of wave dynamics where an additional smoothing effect, in terms of a structural damping, is added to the dynamics of the wave [4], [5]- and references therein. For such models, the wave equation displays enough regularity for standard trace theory to be applicable. On the other end of the spectrum, recent developments in the field for the "real", unperturbed model were focused on only very smooth solutions which lead to topologies where, again, the traces are well defined [7], [8].

In view of the above, a distinct feature of this paper is that we consider *weak* solutions of the structure *without any regularizing effects* for the wave dynamics within finite energy space, where a suitable control problem can be defined. For this problem, we establish existence of global weak solutions in both two and three dimensions. In addition we prove that under additional smoothness of initial conditions, these solutions become strong (classical). We also establish admissibility of a class of boundary controls active on the boundary of the solid. In the case of two dimensions, strong solutions are global and weak solutions are unique, in line with the Navier Stokes theory. In three dimensions, strong solutions are either local in time or are global for sufficiently small initial conditions. Thus, our results recover completely optimal results established for the Navier Stokes equations but in the context of the fluid structure interaction with transmission coupling on the boundary.

As to the mathematical difficulties and challenges, the main obstacle is precisely to acquire a good understanding of trace theory corresponding to weak solutions of the interaction. This necessitates a treatment of the system through microlocal analysis to obtain regularity results of the normal derivative of the wave component on the boundary, which is not readily available from standard trace theory. Only then, one can give a rigorous meaning to weak solutions corresponding to the interaction. In order to establish consistency of weak and strong solutions (for smooth initial data), the main task is to make sense of the role of the pressure term in the equation and the *boundary conditions*. For this step we shall exploit techniques introduced in the analysis of the Navier Stokes equations along with Agmon-Douglis-Nurenberg method that allows for additional regularity of solutions given more regular initial data for the problem.

II. THE MODEL

The mathematical model under consideration is the following. Let $\Omega \in R^n$ be a bounded domain with an interior region Ω_s and an exterior region Ω_f. The boundary Γ_f is the outer boundary of the domain Ω while Γ_s is the boundary of the region Ω_s which also borders the exterior region Ω_f and where the interaction of the two systems take place. Let u be a function defined on Ω_f representing the velocity of the fluid while the scalar function p represents the pressure. Additionally, let w, w_t be the displacement and

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velocity functions of the solid Ω_s. We also denote by ν the unit outward normal vector. The control is represented by $h \in L_2([0.T]; H^{1/2}(\Gamma_s)$ and is active on the boundary Γ_s.

Given any $u \in L_2([0,T]; H^{1/2}(\Gamma_s))$, we are seeking a quadruple (u, w, w_t, p) that satisfy the following system:

$$\begin{cases} u_t - \Delta u + u.\nabla u + \nabla p = 0 & \Omega_f \times [0,T] \\ \text{div} \quad u = 0 & \Omega_f \times [0,T] \\ w_{tt} - \text{div} \quad \sigma(w) = 0 & \Omega_s \times [0,T] \\ u(0,.) = u_0 & \Omega_f \\ w(0,.) = w_0 & \Omega_s \\ w_t(0,.) = w_1 & \Omega_s \\ w_t = u + h & \Gamma_s \times [0,T] \\ u = 0 & \Gamma_f \times [0,T] \\ \sigma(w).\nu = -\frac{\partial}{\partial \nu} u + p\nu + \frac{1}{2}(u.\nu)u & \Gamma_s \times [0,T] \end{cases} \tag{1}$$

We also introduce the energy functional for this system as the following:

$$E(t) = |u(t)|^2 + |\nabla w(t)|^2 + |w_t(t)|^2 \tag{2}$$

III. Notation

Throughout the paper we let $\mathcal{H} \equiv H \times H^1(\Omega_s) \times L_2(\Omega_s)$, where H^1 is topologized with respect to the inner product induced by:

$$(w, z)_1 \equiv \int_\Omega wz + \int_\Omega \sigma(w)\epsilon(z)$$

In addition, we will use the following notation:

$$H \equiv \{u \in L_2(\Omega_f) : \text{div} \;\; u = 0, u.\nu|_{\Gamma_f} = 0\}$$

$$V \equiv \{v \in H^1(\Omega_f) : \text{div}\, v = 0, u|_{\Gamma_f} = 0\}$$

$$\epsilon_{ij}(u) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$$

$$\sigma_{ij}(u) = \lambda(\sum_{k=1}^{k=3} \epsilon_{kk}(u)\delta_{ij} + 2\mu\epsilon_{ij}(u)$$

$$(u, v) = \int_\Omega uv d\Omega$$

$$\langle u, v\rangle = \int_{\Gamma_s} uv d\Gamma_s$$

IV. Weak Solutions for the Uncontrolled Model

In this section we consider weak (variational) solutions to the problem given by (1). We begin with a definition.

Definition 4.1: we say that a triple $(u, w, w_t) \in C([0,T]; H \times H^1(\Omega_s) \times L_2(\Omega_s)) = C([0,T]; \mathcal{H})$ is a weak solution of (1) iff

- $(u(0), w(0), w_t(0)) = (u_0, w_0, w_1) \in H \times H^1(\Omega_s) \times L_2(\Omega_s)$.
- $\sigma(w).\nu \in L_2([0,T]; H^{-1/2}(\Gamma_s))$
- $w_t|_{\Gamma_s} = u|_{\Gamma_s} \in \quad L_2([0,T]; H^{1/2}(\Gamma_s))$
- The following variational equality holds a.e in $t \in (0,T)$

$$\begin{cases} (u_t, \phi) + (\nabla u, \nabla\phi) + (u.\nabla u, \phi) \\ +\langle\sigma(w).\nu, \phi\rangle - \frac{1}{2}\langle(u.\nu)u, \phi\rangle = 0 \\ (w_{tt}, \psi) + (\sigma(w), \epsilon(\psi)) - \langle\sigma(w).\nu, \psi\rangle = 0 \end{cases} \tag{3}$$

for all test functions $\phi \in V$ and $\psi \in H^1(\Omega_s)$

The above definition of a weak solution is very natural. In fact, by projecting the original equation in (1) on H and using boundary conditions, the variational form in (3) is precisely the result of that projection. What is more subtle is to show that any weak solution with sufficient regularity will satisfy the original PDE system. This will be dealt with later on when discussing strong solutions. Our first result pertains to global existence of weak solutions.

Theorem 4.1: Given any initial condition $[u_0, w_0, w_1] \in \mathcal{H}$, there exists a global weak solution (u, w, w_t) to the system 1, i.e. $(u, w, w_t) \in C([0,T]; \mathcal{H})$ such that

$$\begin{aligned} u &\in L_2([0,T]; V) \\ \nabla w|_{\Gamma_s} &\in L_2([0,T]; H^{-1/2}(\Gamma_s)) \\ w_t|_{\Gamma_s} &\in L_2([0,T]; H^{1/2}(\Gamma_s)) \end{aligned} \tag{4}$$

Moreover, in the case when dimension of $\Omega = 2$, weak solutions are unique within the class specified above.

Once existence of weak solutions is established, our next aim is to discuss strong (regular) solutions.

V. Strong Solutions for the Uncontrolled Model

Strong solutions refer to solutions of the original PDE system 1. We begin with a definition.

Definition 5.1: We say that (u, w, w_t, p) is a strong solution of (1) iff:

- $(u, w, w_t, p) \in L_2([0,T]; H^2(\Omega_f) \cap V) \times L_\infty(H^2(\Omega_s) \times H^1(\Omega_s) \times H^1(\Omega_f))$
- $(u_t, w_t, w_{tt}) \in L_\infty([0,T]; H \times H^1(\Omega_s) \times L_2(\Omega_s)) = C([0,T]; \mathcal{H})$
- Strong form of equations given in (1) holds for all $t \in [0,T]$

As expected, in order to be able to obtain strong solutions, one must define compatibility conditions imposed on the initial data. These are given below.

Definition 5.2: We say that initial conditions $(u_0, w_0, w_1) \in V \times H^2(\Omega_s) \times H^1(\Omega_s)$ satisfy Compatibility Conditions (CC) iff

- $w_1 = u_0, \; on \; \Gamma_s$
- $\langle\sigma(w_0)\nu + \frac{\partial}{\partial\nu}u_0 + 1/2(u.\nu)\nu, \phi\rangle = 0$ for all $\phi \in V$.

In order to formulate our results we shall distinguish two and three dimensional domains.

A. The Two Dimensional Case

Theorem 5.1: Let Ω be of dimension 2, then given $[u_0, w_0, w_1] \in H^2(\Omega_f) \bigcap V \times H^2(\Omega_s) \times H^1(\Omega_s)$, such that (CC) conditions hold true, then we have the following:

The corresponding weak solution, asserted by theorem 4.1 becomes a strong solution $[u, w, w_t, p]$ satisfying the system (1) with $p \in C([0,T]; H^1(\Omega_f))$ and such that:

$$u \in C([0,T]; V) \bigcap L_2([0,T]; H^2(\Omega_f)) \quad (5)$$

$$w, w_t \in C([0,T]; H^2(\Omega_s) \times H^1(\Omega_s)) \quad (6)$$

$$p = c(t) \in C([0,T]) \quad on \Gamma_s \quad (7)$$

Moreover, strong solutions are unique.

B. The Three Dimensional Case

In the three dimensional case we shall consider two possible situations. Local -in time- strong solutions and global in time solutions for small initial data.

1) Local in Time Solutions for General Initial Data:

Theorem 5.2: Let Ω be of dimension 3, then given $[u_0, w_0, w_1] \in H^2(\Omega_f) \bigcap V \times H^2(\Omega_s) \times H^1(\Omega_s)$, such that the compatibility condition (CC) 5.2 holds, then there exists $T' > 0$ such that the corresponding weak solution guaranteed by theorem 4.1 becomes the unique strong solution $[u, w, w_t, p]$ satisfying the system (1) and such that:

$$u \in C([0,T']; V) \bigcap L_2([0,T']; H^2(\Omega_f)) \quad (8)$$

$$w, w_t \in C([0,T']; H^2(\Omega_s) \times H^1(\Omega_s)) \quad (9)$$

$$p = c(t) \in C([0,T']) \quad on \quad \Gamma_s. \quad (10)$$

2) Global in Time Solutions for Small Initial Data:

Theorem 5.3: Let Ω be of dimension 3, and let $[u_0, w_0, w_1] \in H^2(\Omega_f) \bigcap V \times H^2(\Omega_s) \times H^1(\Omega_s)$ such that (CC) conditions are satisfied and such that:

$$|u_0|_2^2 + |w_0|_2^2 + |w_1|_1^2 \le \frac{\lambda}{C}$$

where λ, C are constants. Then, there exists a unique strong solution $[u, w, w_t, p]$ satisfying the system (1) for any time T and such that:

$$u \in C([0,T]; V) \bigcap L_2([0,T]; H^2(\Omega_f)) \quad (11)$$

$$w, w_t \in C([0,T]; H^2(\Omega_s) \times H^1(\Omega_s)) \quad (12)$$

$$p = c(t) \in C([0,T]) \quad on \quad \Gamma_s \quad (13)$$

VI. Strategy of the Proof

We shall outline below the main steps of the proof.

Weak solutions. In order to prove existence of weak solutions we shall use a combination of nonlinear semigroup methods along with variational and weak compactness methods. The main steps of the proofs are outlined below.

Step 1: A first step is to consider an ”auxiliary” problem which can be interpreted as a very special truncation of the original problem. Nonlinear semigroup theory will yield global solvability for this auxiliary problem. However, the solution furnished by semigroup theory is the so called ”semigroup” solution. This means that this solution is defined as a strong limit of smooth solutions to an auxiliary problem.

Step 2: The crux of the proof is to show that this semigroup solution satisfies a suitable variational form. It is at this point where the regularity of hyperbolic traces becomes critical. By microlocal analysis methods we are able to secure $H^{-1/2}(\Gamma_s)$ regularity of transmission conditions. The trace regularity result needed is the following:

Theorem 6.1: Consider the linear wave equation.

$$\begin{cases} w_{tt} - \operatorname{div} \sigma(w) = 0 & \Omega_s \times [0,T] \\ w_t|_{\Gamma_s} = g & \in L_2([0,T]; H^{1/2}(\Gamma_s)) \\ w(0,.) = w_0 & \in H^1(\Omega_s) \\ w_t(0,.) = w_1 & \in L_2(\Omega_s) \end{cases} \quad (14)$$

Then the map:

$$(w_0, w_1, g) \to (w, w_t, \sigma(w).\nu)$$

is bounded from:

$$H^1(\Omega_s) \times L_2(\Omega_s) \times L_2([0,T]; H^{1/2}(\Gamma_s))$$

$$\to C([0,T]; H^1(\Omega_s) \times L_2(\Omega_s)) \times L_2([0,T]; H^{-1/2}(\Gamma_s))$$

Step 3: Having obtained a correct variational formulation of semigroup solutions corresponding to the auxiliary problem, we construct the appropriate auxiliary problem for our system by a suitable truncation of the nonlinear term in the Navier Stokes equation.

One has to note that the nonlinear term arising in the variational formulation here involves an extra nonlinear trace term coming from the boundary conditions in addition to the standard nonlinear term of the Navier Stokes. This requires establishing an array of inequalities and properties analogous to those well known for the nonlinear term in the Navier Stokes equations in order to assert the uniform estimates necessary for passing through the limit.

Finally, passing with the limit using weak compactness methods leads to the desired variational formulation of the full nonlinear problem. The proof of uniqueness of weak solution in two dimensions comes via the usual energy estimates and Gronwall's inequality.

Strong Solutions. Strong solutions correspond to the original PDE, hence they involve the pressure term p. Having proved existence of weak solutions, the next step is to analyze regularity of these solutions given the fact that initial data are smoother and that CC conditions in 5.2 are satisfied.

Step 1: We prove that time derivatives $[u_t, w_t, w_{tt}]$ are also well defined in finite energy space $\mathcal{H}$ given the more smooth initial data.

Step 2: In this step we reconstruct the PDE form of the system from the variational form, which is possible given the more regular initial data. In the process of doing this we are also able to conclude that the pressure is constant in space on Γ_s. Reconstructing strong solutions requires a reconstruction of the pressure term which comes via a characterization of the space H and its orthogonal compliment in the L_2 topology. This characterization involves a representation of any element in the compliment of the space H as a gradient of some element in H^1 space with a constant Dirichlet trace on the boundary Γ_s.

$$H^{\perp} = \{v : v = \nabla p, p \in H^1(\Omega_f), p|_{\Gamma_s} = constant\} \quad (15)$$

With this characterization, one utilizes gained regularity of time derivatives of the weak solutions and then reconstructs the original coupled system from the weak form plus the condition that the pressure term must be constant on the boundary Γ_s via the following lemma:

Lemma 6.2: Consider the system:

$$\begin{cases} u_t - \Delta u + \nabla p = f & \Omega_f \times [0,T] \\ \text{div} \quad u = 0 & \Omega_f \times [0,T] \\ w_{tt} - \text{div} \quad \sigma(w) = 0 & \Omega_s \times [0,T] \\ u(0,.) = u_0 & \Omega_f \\ w(0,.) = w_0 & \Omega_s \\ w_t(0,.) = w_1 & \Gamma_s \\ w_t = u & \Gamma_s \times [0,T] \\ u = 0 & \Gamma_f \times [0,T] \\ \sigma(w).\nu = -\frac{\partial}{\partial \nu} u + p + g\nu & \Gamma_s \times [0,T] \end{cases} \quad (16)$$

Let $f \in L_1([0,T]; V')$ and $g \in L_1(0,T; X')$ where:

$$X \equiv \{\phi|_{\Gamma_s}, \phi \in V\} = \{z \in H^{1/2}(\Gamma_s), \int_{\Gamma_s} z \cdot \nu = 0\}$$

If $Y(t) = (u, w, w_t)$ be a weak solution corresponding to (16) and such that $\frac{d}{dt}Y(t) \in C([0,T]; \mathcal{H})$.

Then, there exists $p \in C([0,T]; H^1(\Omega))$ such that $p|_{\Gamma_s} = c(t) \in C([0,T])$, and (u, w, w_t, p) satisfy the strong form in (16).

Step 3: In this step, we aim at obtaining higher space regularity. To accomplish this, we use PDE form of system (guaranteed by Step 2) and we apply Agmon-Douglis-Nurenberg methods. This involves acquiring higher regularity results for the tangential derivatives first. Once that is established, the desired H^2 regularity is attained. Thus, finally we obtain regular (classical) solutions corresponding to the original problem in (1).

VII. The Boundary Control Model

Now that the wellposedness results for the uncontrolled model are in place, we turn our attention to the controlled model. We consider a class of boundary controls $h \in L_2([0,T]; H^{1/2})$ which is compatible with the finite energy topology within which existence of weak solutions is guaranteed by theorem 4.1. We prove the following result pertaining to the controlled model:

Theorem 7.1: Given any control $h \in L_2([0,T]; H^{1/2}(\Gamma_s))$ and any initial condition $[u_0, w_0, w_1] \in \mathcal{H}$, there exists a finite energy solution $[u, w, w_t] \in C([0,T]; \mathcal{H})$ of 1 such that:

$$E(t) \leq CE(0) + C|h|_{L_2([0,T]; H^{1/2}(\Gamma_s))} \quad (17)$$

The idea of the proof of this theorem is to utilize the established regularity of the Neumann trace of the wave component $\sigma w.\nu \in L_2([0,T]; H^{-1/2}(\Gamma_s))$ from theorem 6.1 to estimate the additional trace term $\langle \sigma(w).\nu, h \rangle$ appearing in the variational formulation, hereby obtaining the same wellposedness result as in theorem 4.1 in addition to the estimate in 17.

Remark 7.1: In the linear case, theorem 7.1 asserts that the class of controls $h \in L_2([0,T]; H^{1/2}(\Gamma_s))$ is "admissible" in the language of system theory [17]. This allows for the application of a variety of available results within the theory of admissible controls [1], [11].

VIII. Conclusion

Our work pertaining to this fluid structure interaction model provides a comprehensive wellposedness analysis with sharp regularity results for both weak and strong solutions. Moreover, we establish the admissibility of a certain class of Dirichlet type boundary controls for this model, thus setting the stage for the application of a variety of results from the theory of admissible controls. In our future work, we wish to consider the more physically relevant case of a moving boundary and to also study the asymptotic behavior of the system by providing a complete stability analysis. We also wish to extend our study to include point and Neumann boundary controls in the context of specific optimal control objectives that would be of great interest from the point of view of applications.

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