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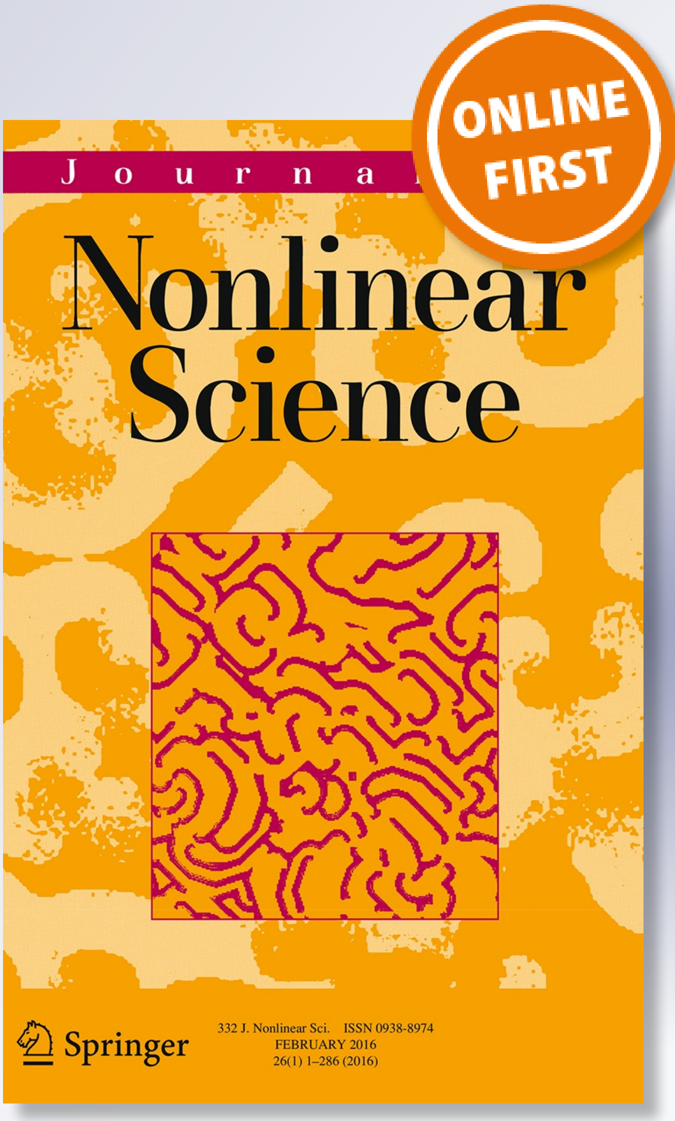

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On Energy Cascades in the Forced 3D Navier–Stokes Equations

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Abstract We show—in the framework of physical scales and (K_1, K_2)-averages—that Kolmogorov's dissipation law combined with the smallness condition on a Taylor length scale is sufficient to guarantee energy cascades in the forced Navier–Stokes equations. Moreover, in the periodic case we establish restrictive scaling laws—in terms of Grashof number—for kinetic energy, energy flux, and energy dissipation rate. These are used to improve our sufficient condition for forced cascades in physical scales.

Keywords Navier–Stokes equations · Turbulence · Energy cascade

Mathematics Subject Classification 35Q30 · 76F02

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1 Introduction

This paper is aimed at a better understanding of mathematical mechanisms behind the phenomenon of energy cascade, one of the main features of turbulence theory dating back to Kolmogorov (1941, 1991). The phenomenological idea behind this theory is that the average energy at any given scale is governed by (i) input from the driving force, (ii) inertial effects that on average transfer energy towards lower scales, and (iii) dissipation due to viscosity. In the fully developed turbulence, beyond the scale of the driving force, τ_f, the inertial effects dominate on a wide range of scales—*the inertial range*—resulting in energy cascading from the higher to lower scales.

Eventually, for very small scales the dissipation becomes dominant, and the energy decays exponentially as the scale R decreases. The threshold where the dissipation effects become noticeable is usually referred to as the *Taylor length scale*, τ_0. The scale at which the dissipation becomes dominant—the *Kolmogorov scale* τ_K—marks the end of the inertial range and the beginning of the *dissipation range*. It is postulated that the energy introduced in the flow by the external force travels through the inertial range only to be completely dissipated inside the dissipation range (this represented schematically in Fig. 1).

Thus, inside the inertial range:

$$\langle \Phi \rangle_R \sim \varepsilon, \tag{1.1}$$

where $\langle \Phi \rangle_R$ is a certain average of the *inward energy flux* Φ at the appropriately defined length scale R and ε is the average *energy dissipation rate*. The above relation can be viewed as the defining feature of the *energy cascade*.

Moreover, this cascade picture is postulated to posses certain universality properties, which imply that the rate of cascade and the above-mentioned Taylor and Kolmogorov length scales depend only on the scale R, viscosity ν, total energy of the flow e, and the energy dissipation rate ε. (Note: assuming that the density is constant, we are considering the energy-related quantities, like e and ε per unit of mass).

Using this phenomenology, one obtains the key laws governing a turbulent flow, particularly, the size of Taylor and Kolmogorov length scales, and the (Kolmogorov) energy spectrum:

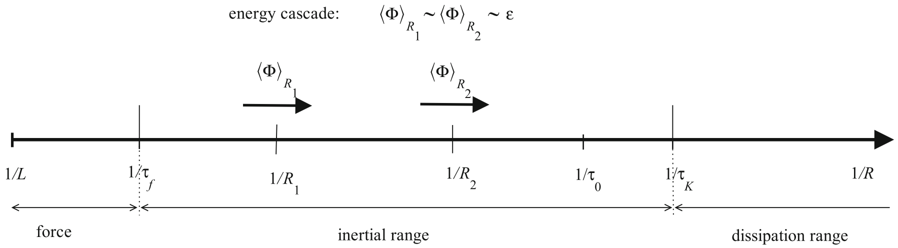

Fig. 1 A schematic representation of an energy cascade: $\langle \Phi \rangle_R$ represents the average inward energy flux through the scale R, ε is the average dissipation rate, while τ_0 and τ_K are, respectively, Taylor and Kolmogorov scales

$$\tau_0 \sim \left(\frac{\nu e}{\varepsilon}\right)^{1/2}, \quad \tau_K \sim \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}, \quad e_R \sim \varepsilon^{2/3} R^{3/2}$$

(in the above, e_R denotes the energy at scale R inside the inertial range).

One of the central relations in the empirical turbulence theory is the so-called Kolmogorov dissipation law. It links the main global parameters of the turbulent flow—the energy dissipation rate ε, the average speed U, and the characteristic length scale L (usually comparable to the linear size of the fluid domain):

$$\varepsilon \sim \frac{U^3}{L}. \tag{1.2}$$

The phenomenological explanation behind the relation above is that the (kinetic) energy $e = U^2/2$ should dissipate in the characteristic time $T = L/U$—the time needed for an average eddy to travel through the inertial range before dissipating. The dissipation law (1.2) was initially discovered experimentally by Richardson (1926) (and phenomenologically in Taylor 1935), and it plays a critical role in our main results (see Theorems 5, 6).

It is important to note that the above relations, although often observed in experiments, have a purely phenomenological explanation, and their derivation does not rely on any equations of motion. For more background on Kolmogorov's empirical theory of turbulence, see e.g. Frisch (1995).

The major challenge of adopting the above phenomenology to mathematically rigorous settings lies in our incomplete understanding of the equations of fluid motion. Additionally, the precise mathematical meaning of the notions of "average" and "length scale" varies. In homogeneous, isotropic turbulence, it is convenient to use the scales provided by Fourier wave numbers (see e.g. Frisch 1995). Moreover, this spectral approach is often used to interpret the experimental data obtained by devices taking measurements on physical scales (Goldstein 1996). Physical scales become important when looking at the geometric structures, e. g. vortices in the turbulent flow. The fact that the empirical laws like (1.1) and (1.2) are confirmed experimentally regardless of the measurement methodology involved points to deeper ergodicity properties of fluid flows, yet to be established mathematically from the underlying equations of motion.

These observations raise a question of whether the phenomenon of energy cascade could be mirrored in the equations of motions, in particular, the 3D Navier–Stokes equations (NSE). This question is also related to the open problem of regularity for the Navier–Stokes system. Indeed, the transfer of energy to lower scales may be connected to a mechanism for a possible finite-time blow-up of the solutions (Katz and Pavlović 2005; Cheskidov 2008; Cheskidov and Friedlander 2009; Tao 2014).

The momentum equation in the incompressible Navier–Stokes system for a fluid inside a domain $\Omega \subseteq \mathbb{R}^3$ can be written as

$$\partial_t u - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f, \tag{1.3}$$

where $t \geq 0$ is the time, $x \in \Omega$, while $u(x, t)$ represents the Eulerian velocity of the fluid, $p(x, t)$ is the pressure (these are subject to appropriate boundary conditions—

e.g. periodic, Dirichlet on a bounded smooth domain, or the whole space with the decay at infinity), and $f(x,t)$ is the external force. The linear term $\nu\Delta u$ represents dissipation and therefore is used to define the energy dissipation rate ε (see Sect. 2). The nonlinearity $(u\cdot\nabla)u+\nabla p$ represents transport of kinetic and potential energy between the scales of the fluid. Consequently, this term is used to define the total energy flux Φ. Thus, mathematically, the question of energy redistribution between the scales of the flow is connected with a careful analysis of the relative magnitudes of these terms in appropriate settings. The presence of the forcing term f further complicates the picture. However, one should note that in the context of many typical boundary conditions (e.g. periodic and Dirichlet), the absence of external force leads to global stability (see Temam 1984), which makes long-time (statistically stationary in Kolmogorov sense) turbulence impossible. Therefore, external forcing (specifically its size as well as its spacial and spectral complexity) plays a particularly important role in the energy cascade formation, that is, currently far from being fully understood.

The first rigorous results about cascade formation in the 3D Navier–Stokes case are obtained in Foias et al. (2001a, 2001b, 2001c). These results use a traditional Fourier approach to scales and propose a formalization of the idea of infinite-time averages by introducing the concept of statistical solutions. A different approach, still connected to the idea of time averages, was employed in Višik and Fursikov (1980). Moreover, in the (more amenable to analysis) 2D case, one can also glimpse into the dependence of solution's long-time behaviour on the nature of the driving force, which, as we noted before, has a direct connection to the question of turbulence (Dascaliuc et al. 2005, 2007; Balci et al. 2010). Another approach to the question of turbulence relies on the Littlewood–Paley decomposition and the use of Besov spaces (Constantin 1997; Cheskidov et al. 2008; Otto and Ramos 2010; Cheskidov and Shvydkoy 2014, see also Sulem and Frisch 1975).

A novel method for the study of turbulent cascades—dubbed (K_1, K_2)-*averaging*—was introduced by the authors in Dascaliuc and Grujić (2011). It uses a form of a "sample" average corresponding to a covering of the domain of interest by balls of size R, satisfying certain optimality conditions [see (2.6), (2.7)]. Thus, the scale in our approach is the actual physical length scale of the domain. The key observation is that near constancy of such averages across *all the possible coverings* at a given scale indicates lack of significant sign fluctuations of the averaged quantity near that particular scale. Mathematically, this approach is amenable to the use of local-in-space estimates on individual balls in the cover, which we connect to global quantities via averaging. It is flexible enough to treat specific subregions of the fluid where Fourier approach is not feasible. Moreover, it can bring spacial complexity and anisotropy into the picture, for example to investigate the effects of vorticity coherence on the enstrophy transfer in 3D, as well as to detect the range of scales of formation and persistence of vortex filaments (Dascaliuc and Grujić 2012c, 2013). As an additional benefit, this approach provides rigorous justification of validity of sampling averaging measurement techniques in turbulent fluids.

The method was utilized in Dascaliuc and Grujić (2011) to obtain a set of sufficient conditions for energy cascades (for decaying turbulence) in the unforced NSE, which are consistent with the ones obtained in Foias et al. (2001b, 2001c), as well as to establish locality of such cascades. Scale locality is still an open question in the

Fourier setting, although important progress has been made in Cheskidov et al. (2008) where the authors established quasi-locality of the flux in the Littlewood–Paley setting (see also Eyink 2005 for a different approach to locality). Another instance in which (K_1, K_2)-averaging proved fruitful was our study of the dissipation anomaly, where, in the inviscid limit, the Navier–Stokes solutions displaying the ever wider inertial ranges converge to a "dissipative" (i.e. manifesting anomalous energy dissipation) solution of the Euler equations, for which the energy cascade continues *ad infinitum* (Dascaliuc and Grujić 2012a, b).

The present paper extends the results of Dascaliuc and Grujić (2011) to the forced case. The body force presents a problem due to the lack of adequate small-scale estimates on the localized forcing term $f \cdot u$. Nevertheless, we establish a set of sufficient conditions for the energy cascade on a localized region inside the domain Ω, which can be applied regardless of the boundary conditions, as long as the suitable weak solutions exist—see Theorem 1. However, in addition to the condition on the smallness of the Taylor scale present in Dascaliuc and Grujić (2011), we need to assume a relation involving the size of the force and the energy that does not hold globally on Ω in certain boundary settings, particularly, in the space-periodic case (see Remark 3). To overcome this deficiency, we set on to exploit the additional structure offered by space periodicity in order to obtain a better bound of the aforementioned $f \cdot u$-term in Sects. 4 and 5.

The key observation is that in the space-periodic case, the dissipation law (1.2) leads to a restrictive set of scaling properties for both energy e and its dissipation rate ε (see Theorem 5 and Remark 4). Namely, ε scales like $Gr^{3/2}$, while the kinetic energy scales like Gr, where Gr is the Grashof number, representing non-dimensional magnitude of the force (2.19), (2.25). These types of scaling laws were discovered in the context of statistical solutions in Dascaliuc et al. (2009), and the fact that *the same scaling* is manifested in the framework of (K_1, K_2)-averages points to a remarkable consistency between the two approaches. We exploit this scaling to obtain a saturation property for Cauchy–Schwarz-type inequality for averages of $f \cdot u$, which, in turn, allows us to formulate a sufficient condition for energy cascades inside the entire domain Ω, as well as to estimate the width of the inertial range in terms of Grashof number (see Theorem 6 and the remarks thereafter).

We finish the paper by pointing out several links between the notions of suitable and Leray–Hopf weak solutions in order to make the connection between the approach using Fourier scales and statistical solutions (infinite-time averages), and our approach using physical scales and (K_1, K_2)-averages more evident.

2 Preliminaries

2.1 Suitable Weak Solutions

Consider 3D incompressible NSE in a certain domain $\Omega \subseteq \mathbb{R}^3$ driven by an external body force f

$$\begin{aligned} \partial_t u - \nu \Delta u + (u \cdot \nabla) u + \nabla p &= f \\ \nabla \cdot u &= 0 \end{aligned} \tag{2.1}$$

For the most part, we will consider *suitable weak solutions* of (2.1), the notion first introduced by Scheffer (see Scheffer 1977; Caffarelli et al. 1982; Lemarié-Rieusset 2002). These are weak (distributional) solutions that satisfy in particular the local energy inequality:

$$\iint \left(\frac{|u|^2}{2} + p\right) u \cdot \nabla\phi \geq \nu \iint |\nabla u|^2\phi - \iint \frac{|u|^2}{2}\left(\partial_t\phi + \nu\Delta\phi\right) - \iint f \cdot u\,\phi \tag{2.2}$$

holds for any C_0^∞-test function $\phi(t, x) \geq 0$.

The suitable solutions are known to exist in many standard scenarios, including the $\Omega = \mathbb{R}^3$ (with decay at infinity), space periodic, and other typical boundary conditions. Notably, the problem of uniqueness and regularity are still open questions in all these settings, although via a well-known regularity result, the one-dimensional Hausdorff measure of space–time singular points of such solution is zero (Caffarelli et al. 1982).

2.2 (K_1, K_2)-Averages

To study the behaviour of physical quantities in the context of suitable solutions of (2.1), we will employ (K_1, K_2)-averages defined as follows.

Let K_1 and K_2 be the fixed positive constants and let Ω_0 be a subdomain of Ω. We generally require Ω_0 to be a bounded simply-connected domain with a piecewise-smooth boundary with the distance from $\partial\Omega$ to $\partial\Omega_0$ to be bigger then or comparable the diameter of Ω_0. For simplicity, assume

$$\Omega_0 = B(0, R_0) \quad \text{for some} \quad R_0 > 0 \quad \text{and} \quad B(0, 2R_0) \subseteq \Omega. \tag{2.3}$$

We will refer to Ω_0 as *integral domain*, and to its radius $R_0 > 0$ as the *integral scale*.

To localize the solutions in space and time, we will employ the *refined cut-off functions*:

Let $\delta \in (1/2, 1)$ be fixed and $C_0 (= C_0(\delta))$ be a sufficiently large positive constant. For each ball $B(x_0, R)$, a refined space cut-off is a function $\psi(x)$ such that $0 \leq \psi \leq 1$, $\mathrm{Supp}(\psi) \subseteq B(x_0, 2R)$, $\psi = 1$ on $B(x_0, R)$, and

$$|\nabla\psi(x)| \leq \frac{C_0}{R}\,\psi^\delta(x), \qquad |\nabla\psi(x)| \leq \frac{C_0}{R^2}\,\psi^{2\delta-1}(x). \tag{2.4}$$

A refined time cut-off on $[0, T]$ is a function $\eta(t)$ such that $0 \leq \eta \leq 1$, $\mathrm{Supp}(\eta) \subseteq [0, T]$, and

$$\left|\eta'(x)\right| \leq \frac{C_0}{T}\,\eta^\delta(x). \tag{2.5}$$

The existence of such refined cut-off functions was noted in Zhang (2004).

Now, let $0 < R < R_0$. Define a (K_1, K_2)-*covering at scale* R any covering of Ω_0 by n balls of radius R, $\{B(x_i, R)\}_{i=\overline{1,n}}$, such that: each $x_i \in \Omega_0$ and

$$\left(\frac{R_0}{R}\right)^3 \leq n \leq K_1 \left(\frac{R_0}{R}\right)^3; \tag{2.6}$$

$$\text{each point in } \Omega_0 \text{ is covered by no more then } K_2 \text{ balls } B(x_i, 2R). \tag{2.7}$$

We call K_1 and K_2, respectively, the *global* and *local multiplicities* of the covering. Note that such coverings exist if K_1 and K_2 are large enough. Moreover, K_2 in fact determines an upper bound on n (the lower bound is determined by the geometry of $\mathbb{R}^3$). For example, we can choose $K_1 = K_2 = 8$.

Let ψ_0 be a fixed global cut-off, i.e. a refined cut-off associated with Ω_0.

Given, a (K_1, K_2)-covering $\{B(x_i, R)\}_{i=\overline{1,n}}$, associate each ball $B(x_i, R)$ a refined cut-offs ψ_i. Notice that for all $x \in \Omega_0$,

$$\psi_0(x) \leq \sum_{i=1}^{n} \psi_i(x) \leq K_2 \psi_0(x) \tag{2.8}$$

In order to ensure compatibility of boundary elements with the global cut-off ψ_0, we will change the support assumptions for the cut-offs on the boundary of Ω_0 so that (2.8) actually holds for all $x \in \mathbb{R}^3$, while still satisfying bounds (2.4).

If Q is a physical quantity with the density q (or more general, if q is a distribution in $\mathcal{D}'(\Omega \times [0, T])$), then define a the (K_1, K_2)-*average of* Q *at scale* R *associated with* $\{\psi_i\}_{i=\overline{1,n}}$ the quantity

$$\langle Q \rangle_R = \frac{1}{n} \frac{1}{R^3} \frac{1}{T} \sum_{i=1}^{n} (q, \eta \psi_i). \tag{2.9}$$

We will note that if q is a non-negative distribution, then all the averages on all scales are comparable to the global (integral scale) average:

$$\frac{1}{K_1} Q_0 \leq \frac{1}{n} \left(\frac{R_0}{R}\right)^3 Q_0 \leq \langle Q \rangle_R \leq K_2 \frac{1}{n} \left(\frac{R_0}{R}\right)^3 Q_0 \leq K_2 Q_0, \tag{2.10}$$

where the global average Q_0 is

$$Q_0 = \frac{1}{R_0{}^3} \frac{1}{T} (q, \eta \phi_0) . \tag{2.11}$$

In the case of a sign-changing distribution, the averages can differ widely from scale to scale and depend largely the particular choice of the (K_1, K_2)-covering as well as cut-offs ψ_i. If for a range of scales *all* the averages are comparable, it means that in statistical sense there are no significant fluctuations of the sign of q on those scales. However, the averages may became uncorrelated on smaller scales.

For example, in 1D for $K_1 = K_2 = 3$, $N \gg 1$, if $q = M(0.5 + \sin(Nx))$ on an interval $[-\pi, \pi]$ (i.e. $R_0 = \pi$), the global average is $Q_0 = M/2$, and on scales $R \gg 1/N$ (e.g. $R > 10/N$), all the averages will be comparable to Q_0. However, if $R < 1/(4N)$, the averages depend in an essential way on the choice of the covering: if we stack the balls in the covering to emphasize negative areas of q, the averages will be comparable to $-Q_0 \sim -M$, while if we focus on positive areas we obtain averages comparable to $Q_0 \sim M$. Of course, we can also obtain anything in between. Note that the scale on which q changes sign is $\sim 1/N$.

Let $\phi = \eta\psi$, where ψ is a refined cut-off corresponding to $B(x_0, R)$. In the framework of turbulence theory, we distinguish the following physical quantities (all per unit of mass on $B(x_0, R)$ over time interval $[0, T]$):

$$e_{x_0,R,T} = \frac{1}{T}\frac{1}{R^3}\iint \frac{|u|^2}{2}\phi^{2\delta-1} \quad \text{localized kinetic energy,}$$

$$\varepsilon_{x_0,R,T} = \epsilon^{\infty}_{x_0,R,T} + \frac{1}{T}\frac{1}{R^3}\nu\iint |\nabla u|^2\phi \quad \text{localized viscous + anomalous energy dissipation,}$$

$$\Phi_{x_0,R,T} = \frac{1}{T}\frac{1}{R^3}\iint \left(\frac{|u|^2}{2} + p\right) u\cdot\nabla\phi \quad \text{kinetic + potential inward energy flux,}$$

$$|f|^2_{x_0,R,T} = \frac{1}{n}\frac{1}{T}\frac{1}{R^3}\iint |f|^2\phi \quad \text{localized square of the force.}$$

(the powers associated with ϕ are chosen for technical reasons—see the Sect. 3).

In the notations above, the local energy inequality (2.2) becomes

$$\Phi_{x_0,R,T} = \varepsilon_{x_0,R,T} - \frac{1}{T}\frac{1}{R^3}\iint \frac{|u|^2}{2}(\partial_t\phi + \nu\Delta\phi) - \frac{1}{T}\frac{1}{R^3}\iint f\cdot u\,\phi. \tag{2.12}$$

In particular, the equality in the above is achieved by adding the anomalous dissipation $\epsilon^{\infty}_{x_0,R,T}$—technically, a non-negative distribution ϵ^{∞} evaluated at ϕ.

Remark 1 While the motivation behind the above definitions for the localized energy and its dissipation rate are transparent, the definition of the energy flux merits an explanation. In the context of regular (smooth) solutions, the energy balance on a ball $B_R = B(x_0, R) \subset \Omega$ reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{B_R}|u|^2 = \nu\int_{B_R}\Delta u\cdot u \;-\int_{B_R}[(u\cdot\nabla)u + \nabla p]\cdot u + \int_{B_R} f\cdot u\,.$$

Using integration by parts in the second term of the right-hand side, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{B_R}|u|^2 = \nu\int_{B_R}\Delta u\cdot u \;+\int_{\partial B_R}\left(\frac{|u|^2}{2} + p\right) u\cdot n + \int_{B_R} f\cdot u\,, \tag{2.13}$$

where $n(x)$ is the *inward* normal at $x \in \partial B_R$. Note that in Physics, the term $\int_{\partial B_R}\left[(|u|^2/2) + p\right]\, u\cdot n$ is the *classical* flux of the quantity $(|u|^2/2) + p$ through the boundary ∂B_R *inside* the ball B_R. In particular, (2.13) implies that if the sign of the flux term is positive, it contributes to the increase in the (kinetic) energy inside the ball due to the inflow of kinetic and potential energy through the boundary (of course the total balance is also affected by the dissipation $\nu \int_{B_R} \Delta u \cdot u$ as well as forcing $\int_{B_R} f \cdot u$). In the case of $\Phi_{x_0,R,T}$, the cut-off function $\phi(x,t)$ can be chosen to decrease radially outside $B(x_0, R)$, so that $\nabla\phi$ and n have the same direction. Consequently, $\Phi_{x_0,R,T}$ can be interpreted as a space–time average of the *inward flux of energy* into B_R *through the shell* between $B(x_0, 2R)$ and $B(x_0, R)$. Note that if we choose to localize the energy via $\phi(x,t) = \eta(t)\psi(x)$ rather then by using a discontinuous cut-off, we obtain

$$\eta(t)\,\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int\limits_{B_{2R}} |u|^2\psi = \nu \int\limits_{B_{2R}} \Delta u \cdot u\,\phi \;+ \int\limits_{B_{2R}\setminus B_R} \left(\frac{|u|^2}{2} + p\right) u \cdot \nabla\phi + \int\limits_{B_{2R}} f \cdot u\,\phi\,, \tag{2.14}$$

thus confirming that the *positivity* of $\Phi_{x_0,R,T}$ contributes to the growth in kinetic energy inside the ball (on average over the time T). Therefore, we will interpret the (K_1, K_2)-average of $-[(u\cdot\nabla)u + p]\cdot u$ (see $\langle\Phi\rangle_R$ below) as the average of the *inward energy flux through the scale* R, its positivity signalling that, at the scale R over the time interval $[0, T]$, the energy tends to flow to lower scales—a signature feature of a turbulent energy cascade.

Given a (K_1, K_2)-cover of Ω_0 and a corresponding collection of refined cut-offs $\{\phi_i\}_{i=\overline{1,n}}$, we have (K_1, K_2)-averages of energy, dissipation rate, flux, and the square of the force:

$$\langle e\rangle_R = \frac{1}{n}\sum_{i=1}^{n} e_{x_i,R,T}, \quad \langle\varepsilon\rangle_R = \frac{1}{n}\sum_{i=1}^{n} \varepsilon_{x_i,R,T},$$
$$\langle\Phi\rangle_R = \frac{1}{n}\sum_{i=1}^{n} \Phi_{x_i,R,T}, \quad \langle|f|^2\rangle_R = \frac{1}{n}\sum_{i=1}^{n} |f|^2_{x_0,R,T}.$$

Remark 2 (a) Strictly speaking $\langle e\rangle_R$ is not the average of energy density $|u|^2/2$ due to the $(2\delta - 1)$-power attached to ϕ_i. Nevertheless, this average enjoys the same positivity property as the usual averages defined by (2.9).

(b) The other averages—$\langle\varepsilon\rangle_R$, $\langle\Phi\rangle_R$, and $\langle|f|^2\rangle_R$, are precisely the (K_1, K_2)-averages of the corresponding distributions in the sense of (2.9). Moreover, notice that the ε and $|f|^2$ are non-negative distributions, while Φ is a sign-varying distribution $-[(u\cdot\nabla)\,u + \nabla p]\cdot u$.

We also define the global space–time averages

$$e_0 = \frac{1}{T}\frac{1}{R_0^3}\iint \frac{|u|^2}{2}\phi_0^{2\delta-1}, \quad \varepsilon_0 = \epsilon_0^\infty + \frac{1}{T}\frac{1}{R_0^3}\nu\iint |\nabla u|^2\phi_0,$$
$$\Phi_0 = \frac{1}{T}\frac{1}{R_0^3}\iint \left(\frac{|u|^2}{2} + p\right) u\cdot\nabla\phi_0, \quad |f|_0^2 = \frac{1}{T}\frac{1}{R_0^3}\iint |f|^2\phi_0. \tag{2.15}$$

where $\phi_0 = \eta\psi_0$ is the global cut-off and ϵ_0^∞ is the anomalous dissipation corresponding to Ω_0.

On one hand, consistently with the positivity property (2.10),

$$\frac{1}{K_1} e_0 \le \frac{1}{n}\left(\frac{R_0}{R}\right)^3 e_0 \le \langle e\rangle_R \le K_2 \frac{1}{n}\left(\frac{R_0}{R}\right)^3 e_0 \le K_2 e_0. \tag{2.16}$$

$$\frac{1}{K_1} \varepsilon_0 \le \frac{1}{n}\left(\frac{R_0}{R}\right)^3 \varepsilon_0 \le \langle \varepsilon\rangle_R \le K_2 \frac{1}{n}\left(\frac{R_0}{R}\right)^3 \varepsilon_0 \le K_2 \varepsilon_0, \tag{2.17}$$

end

$$\frac{1}{K_1} |f|_0^2 \le \frac{1}{n}\left(\frac{R_0}{R}\right)^3 |f|_0^2 \le \langle |f|^2\rangle_R \le K_2 \frac{1}{n}\left(\frac{R_0}{R}\right)^3 |f|_0^2 \le K_2 |f|_0^2. \tag{2.18}$$

On the other hand, Φ is an a priory sign-varying quantity, and by the above discussion, we will interpret the positivity of $\langle \Phi\rangle_R$ over a range of scales R as the fact that, in average, the energy flows towards the lower scales over that range of scales—i.e. *energy cascade* takes place.

In relation to turbulence, it is convenient to use adimensional parameters as an indicator of complexity of the flow. In particular, we will use *Grashof number*, Gr, which characterizes the average magnitude of the driving force and the *Reynolds number*, Re, which characterizes the magnitude of the average velocity:

$$Gr = \frac{(|f|_0^2)^{1/2}}{\nu^2 R_0^{-3}}, \quad Re = \frac{e_0^{1/2}}{\nu R_0^{-1}}. \tag{2.19}$$

Note that the expressions for Gr and Re are in fact the adimensional expressions for localized (to $\Omega_0 \times [0, T]$) L^2-norms of f and u, respectively. For convenience, we will also denote by F_0 the root-mean-square average of the force:

$$F_0 = (|f|_0^2)^{1/2} = \left(\frac{1}{T}\frac{1}{R_0^3}\iint |f|^2 \phi_0\right)^{1/2}. \tag{2.20}$$

2.3 Leray–Hopf Solutions in Periodic Case

While the main result of Sect. 3 is valid for any suitable weak solution, the last two sections of this paper will concentrate on space-periodic flows. In this case, we have a convenient functional formulation for (2.1). We refer to Temam (1984, 1997) for more background on the settings described below.

Let $\Omega = [0, L]^3$, $L = 4R_0$, while $f \in L^3(\Omega)$ is an Ω-periodic, divergence-free, *time-independent* force.

By changing coordinates, we can consider that the solutions u and the force f have zero averages, i.e.

$$\int\limits_{[0,L]^3} u = \int\limits_{[0,L]^3} f = 0.$$

Define

$$H = \{u \in L^2(\omega) \ : \ u \text{ is } \Omega\text{-periodic}, \ \nabla \cdot u = 0, \int_\Omega u = 0\},$$

and denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the L^2-norm and L^2-inner product on Ω.

For notational convenience, let $A = -\Delta$—the Stokes operator, and $B(u,v) = P_L(u\cdot\nabla)u$, where P_L is the Leray projector. Note that $(B(u,v),w) = \int_{[0,L]^3}(u\cdot\nabla v)\cdot w$, while A is a positive-definite, self-adjoint unbounded operator with a compact (in L^2) inverse, and thus, the powers of A are well defined. Moreover, the norms $\|A^{\alpha/2}\cdot\|^2$ are equivalent to H^α-Sobolev norms on Ω. We will denote $V = D(A^{1/2})$, and V'-the dual of V.

Then we can write NSE in the following functional form on H:

$$u_t + \nu A u + B(u,u) = f \in H. \tag{2.21}$$

In this case, one can prove existence of *Leray–Hopf weak solutions*, i.e. $u \in L^\infty([0,\infty),H) \cap L^2_{loc}((0,\infty),V)$, such that (2.21) is satisfied in V':

$$(u,v)_t + (A^{1/2}u, A^{1/2}v) - (B(u,v),u) = (f,v) \quad \text{in distribution sense for all } v \in V, \tag{2.22}$$

and for which the *Leray–Hopf energy inequality* holds:

$$\nu \int\limits_{t_0}^{t_1} \|\nabla u\|^2 \leq \frac{\|u(t_0)\|^2}{2} - \frac{\|u(t_1)\|^2}{2} + \int\limits_{t_0}^{t_1} (f,u) \qquad \text{a.e. in } t_0 \text{ and for all } t_1 \geq t_0. \tag{2.23}$$

In particular, the Leray–Hopf solutions are regular (i.e. in $D(A)$) for all times, except possibly on a closed set of one-dimensional Hausdorff measure zero. Therefore, (2.23) becomes equality on a dense set of disjoint open time intervals in $[0,\infty)$.

Another peculiar feature of Leray–Hopf solutions is the existence of the *weak global attractor*, $\mathcal{A}_w$ (cf. Foias and Temam 1987; Cheskidov 2009; Foias et al. 2010). The set $\mathcal{A}_w \subset H$ can be defined as the set of all global (forward and backwards) in time Leray–Hopf solutions. One can show that $\mathcal{A}_w$ is bounded in H and weakly attracts bounded sets as $t \to \infty$.

It is known (see e.g. Lemarié-Rieusset 2002) that the solutions originally constructed by Leray (see Leray 1934) will satisfy both local energy inequality (2.2) [i.e. they are suitable solutions in Scheffer (1977) and Caffarelli et al. (1982) sense] as well as the global Leray energy inequality (2.23).

We note that the argument given in Lemarié-Rieusset (2002) is in whole $\mathbb{R}^3$; however, similar result holds in the periodic case, in fact one can prove that Leray's method provides solutions that satisfy the following version of the Leray–Hopf energy inequality:

$$\nu \int_0^T \eta \, \|\nabla u\|^2 \leq \int_0^T \eta_t \, \frac{\|u\|^2}{2} + \int_0^T \eta \, (f, u), \tag{2.24}$$

where η is a refined cut-off on $[0, T]$ provided by (2.5).

In these settings, we will find more convenient to use a slightly modified—non-localized—version of Grashof number:

$$Gr = \frac{\|f\|}{\nu^2 R_0^{-3/2}}. \tag{2.25}$$

As we show in "Appendix", *any* Leray–Hopf solution will actually satisfy (2.24). Moreover, *any suitable weak solution* in the periodic case in addition to the local energy inequality (2.2) will necessarily satisfy (2.24) as well.

Thus, in Sect. 5, when talking about periodic solutions, we will assume that we are dealing with solutions that satisfy both local and global energy inequalities (2.2) and (2.24). We note that the results of Sect. 4 do not require that solutions are suitable in Scheffer (1977) sense, i.e. (2.2) is not assumed.

3 Energy Cascade Theorem

Assume $f \in L^2(\Omega \times [0, T])$ is a driving force in the context of suitable solutions to NSE [see (2.2)]. Also, let $\{\phi_i\}_{i=\overline{1,n}}$, $\phi_i = \eta\psi_i$, be a family of refined cut-offs corresponding to a (K_1, K_2)-average (see Sect. 2.2).

For convenience, assume

$$T \geq \frac{R_0^2}{\nu}. \tag{3.1}$$

Also denote

$$e_i = e_{x_i,R,T}, \quad \varepsilon_i = \varepsilon_{x_i,R,T}, \quad \text{and} \quad \Phi_i = \Phi_{x_i,R,T}.$$

Then the local energy balance [see (2.12)] is

$$\Phi_i = \varepsilon_i - \frac{1}{T}\frac{1}{R^3} \iint \frac{|u|^2}{2} \left(\partial_t \phi_i + \nu \Delta \phi_i\right) - \frac{1}{T}\frac{1}{R^3} \iint f \cdot u \, \phi_i. \tag{3.2}$$

Recall, (2.17) states:

$$\frac{1}{n}\left(\frac{R_0}{R}\right)^3 \varepsilon_0 \leq \langle \varepsilon \rangle_R \leq K_2 \frac{1}{n}\left(\frac{R_0}{R}\right)^3 \varepsilon_0. \tag{3.3}$$

Use (2.4) and (2.5) to estimate:

$$\left|\frac{1}{T}\frac{1}{R^3}\iint\frac{|u|^2}{2}\,(\partial_t\phi_i+\nu\Delta\phi_i)\right|\le C_0\left(\frac{1}{T}+\frac{\nu}{R^2}\right)\frac{1}{T}\frac{1}{R^3}\iint\frac{|u|^2}{2}\eta^\delta\psi_i^{2\delta-1}$$
$$\le 2\,C_0\frac{\nu}{R^2}\,e_i\,, \tag{3.4}$$

and thus, by (2.16),

$$\left|\left\langle\iint\frac{|u|^2}{2}\,(\partial_t\phi_i+\nu\Delta\phi_i)\right\rangle_R\right|\le 2\,C_0K_2\frac{\nu}{R^2}\,\frac{1}{n}\left(\frac{R_0}{R}\right)^3 e_0. \tag{3.5}$$

We bound the force term by:

$$\left|\frac{1}{T}\frac{1}{R^3}\iint f\cdot u\,\phi_i\right|\le\sqrt{2}\frac{1}{T}\frac{1}{R^3}\left(\iint|f|^2\phi_i^{3-2\delta}\right)^{1/2}\left(\iint\frac{|u|^2}{2}\phi_i^{2\delta-1}\right)^{1/2}$$
$$=\sqrt{2}\left(\frac{1}{T}\frac{1}{R^3}\iint|f|^2\phi_i^{3-2\delta}\right)^{1/2}\,e_i^{1/2},$$

and consequently

$$\begin{aligned}
|\langle\,(f,u)\,\rangle_R|&\le\sqrt{2}\,\frac{1}{n}\sum_{i=1}^n\left(\frac{1}{T}\frac{1}{R^3}\iint|f|^2\phi_i^{3-2\delta}\right)^{1/2}\,e_i^{1/2}\\
&\le\sqrt{2}\frac{1}{n}\left(\frac{1}{T}\frac{1}{R^3}\sum_{i=1}^n\iint|f|^2\phi_i^{3-2\delta}\right)^{1/2}\left(\sum_{i=1}^n e_i\right)^{1/2}\\
&\le\sqrt{2}\left(\frac{1}{n}\frac{1}{T}\frac{1}{R^3}\sum_{i=1}^n\iint|f|^2\phi_i^{3-2\delta}\right)^{1/2}\left(\frac{1}{n}\sum_{i=1}^n e_i\right)^{1/2}\\
&\le\sqrt{2}\,\langle|f|^2\rangle_R^{1/2}\langle e\rangle_R^{1/2},
\end{aligned}$$

where in the last inequality we use that $\phi_i^{3-2\delta}\le\phi_i$ (since $3-2\delta>1$). Thus, by (2.18), we have

$$\langle|f|^2\rangle_R\le K_2\,\frac{1}{n}\left(\frac{R_0}{R}\right)^3 F_0^2,$$

and consequently,

$$|\langle\,(f,u)\,\rangle_R|\le\sqrt{2}\,K_2\,\frac{1}{n}\left(\frac{R_0}{R}\right)^3 F_0\,e_0^{1/2}. \tag{3.6}$$

Next, taking the average in (3.2), using the bounds (3.3), (3.4), and (3.6), we obtain:

$$\frac{1}{n}\left(\frac{R_0}{R}\right)^3\left[\varepsilon_0 - 2C_0K_2\frac{\nu}{R^2}\,e_0 - \sqrt{2}K_2F_0\,e_0^{1/2}\right] \le \langle\Phi\rangle_R$$
$$\le K_2\,\frac{1}{n}\left(\frac{R_0}{R}\right)^3\left[\varepsilon_0 + 2C_0\frac{\nu}{R^2}\,e_0 + \sqrt{2}F_0\,e_0^{1/2}\right]. \tag{3.7}$$

Notice that (2.6) implies

$$\frac{1}{K_1} \le \frac{1}{n}\left(\frac{R_0}{R}\right)^3 \le 1, \tag{3.8}$$

and so we obtain

$$\frac{1}{K_1}\left[\varepsilon_0 - 2C_0K_2\frac{\nu}{R^2}\,e_0 - \sqrt{2}K_2F_0\,e_0^{1/2}\right] \le \langle\Phi\rangle_R$$
$$\le K_2\left[\varepsilon_0 + 2C_0\frac{\nu}{R^2}\,e_0 + \sqrt{2}F_0\,e_0^{1/2}\right]$$

Assume:

$$\sqrt{2}\,K_2\,F_0\,e_0^{1/2} \le \frac{1}{2}\varepsilon_0\,,$$

Then

$$\frac{1}{K_1}\left[\frac{1}{2}\varepsilon_0 - 2C_0K_2\frac{\nu}{R^2}e_0\right] \le \langle\Phi\rangle_R \le K_2\left[\frac{3}{2}\varepsilon_0 + 2C_0\frac{\nu}{R^2}e_0\right],$$

which implies

$$\frac{1}{2K_1}\varepsilon_0\left(1 - 4C_0K_2\frac{\nu e_0/\varepsilon_0}{R^2}\right) \le \langle\Phi\rangle_R \le \frac{3K_2}{2}\varepsilon_0\left(1 + \frac{4C_0}{3}\frac{\nu e_0/\varepsilon_0}{R^2}\right),$$

and consequently,

$$\frac{1}{2K_1}\varepsilon_0\left(1 - \alpha\frac{\tau_0^2}{R^2}\right) \le \langle\Phi\rangle_R \le \frac{3K_2}{2}\varepsilon_0\left(1 + \alpha\frac{\tau_0^2}{R^2}\right), \tag{3.9}$$

where

$$\alpha = 4C_0K_2 \tag{3.10}$$

and

$$\tau_0 = \left(\frac{\nu e_0}{\varepsilon_0}\right)^{1/2} \quad \text{is Taylor scale.} \tag{3.11}$$

If $\alpha\tau_0^2/R_0^2 < 1/2$ (or, equivalently $8C_0K_0\,\nu e_0/R_0^2 \le \varepsilon_0$), the inequality above implies the following theorem:

Theorem 1 *Let* (3.1) *be satisfied and denote*

$$\alpha_0 = \sqrt{8}K_2 \quad \textit{and} \quad \beta_0 = 8C_0K_2. \tag{3.12}$$

Assume

$$\alpha_0 F_0\, e_0^{1/2} \le \varepsilon_0, \tag{3.13}$$

and

$$\beta_0 \nu \frac{e_0}{R_0^2} \le \varepsilon_0 \tag{3.14}$$

hold. Then we have energy cascade

$$\frac{1}{4K_1}\,\varepsilon_0 \le \langle \Phi \rangle_R \le \frac{9K_2}{4}\,\varepsilon_0 \tag{3.15}$$

for all R inside the inertial range

$$\left[\sqrt{\beta_0}\,\tau_0,\ R_0\right]. \tag{3.16}$$

Notice that the condition (3.14) is essentially the same as in the cascade theorem for the unforced case, see Dascaliuc and Grujić (2011). It requires that Taylor length scale τ_0 is smaller then the integral scale R_0. The other condition, (3.13), contains dependence on the size of the force (automatically satisfied if $f = 0$), which requires the energy dissipation rate to be big enough compared to both the magnitude of f and e_0. As we will see in the periodic case, this condition amounts to saturation of Kolmogorov dissipation law. Note that the width of the inertial range (3.16) is not directly dependent on (3.13).

4 Bounds on Average Energy and Enstrophy

Consider the 3D incompressible NSE with periodic boundary conditions inside $\Omega = [0, L]^3$, driven by an L^2, divergence-free, time-independent force f—see Sect. 2.3. Set the integral scale $R_0 = L/4$.

Our goal is to study the kinetic energy and its dissipation rate on the *whole* periodic box Ω. As we noted before, in this section we will only require that u is a Leray–Hopf solution and as consequence (see Theorem 8 and the Remark that follows) satisfy the energy inequality (2.24). Thus, the results would hold even for (hypothetical) Leray–Hopf solutions that are not suitable in Scheffer sense (Scheffer 1977).

4.1 A Consequence of the Energy Inequality

We can take advantage of periodicity to simplify definitions for global energy and enstrophy used in Sects. 2.2 and 3.

Namely, define the (modified) global energy

$$e_0 = \frac{1}{T}\frac{1}{R_0^3}\int\limits_0^T \eta^\delta \frac{\|u\|^2}{2}, \tag{4.1}$$

and the total (viscous+anomalous) dissipation rate:

$$\varepsilon_0 = \frac{1}{T}\frac{1}{R_0^3}\nu\int\limits_0^T \eta\,\|\nabla u\|^2 + \varepsilon_\infty, \tag{4.2}$$

where the anomalous dissipation $\epsilon_\infty \geq 0$ is such that

$$\varepsilon_0 = \frac{1}{T}\frac{1}{R_0^3}\int\limits_0^T \eta_t \frac{\|u\|^2}{2} + \frac{1}{T}\frac{1}{R_0^3}\int\limits_0^T \eta\,(f,u). \tag{4.3}$$

We bound

$$\left|\frac{1}{T}\frac{1}{R_0^3}\int_0^T \eta_t \frac{\|u\|^2}{2}\right| \leq \frac{C_0}{T}e_0, \left|\frac{1}{T}\frac{1}{R_0^3}\int_0^T \eta\,(f,u)\right|$$
$$\leq \frac{1}{T}\frac{1}{R_0^3}\left(\int_0^T \eta^{2-\delta}\|f\|^2\right)^{1/2}\left(\int\limits_0^T \eta^\delta\|u\|^2\right)^{1/2} \leq \sqrt{2}\frac{\|f\|}{R_0^{3/2}}e_0^{1/2}.$$

Use these in (4.3) to obtain

$$\varepsilon_0 \leq C_0\frac{e_0}{T} + \sqrt{2}\frac{\|f\|}{R_0^{3/2}}e_0^{1/2} \tag{4.4}$$

If $u \in \mathcal{A}_w$, i.e. u is on the weak attractor (cf. Foias and Temam 1987), then

$$\|u\|^2 \leq \left(\frac{2}{\pi}\right)^4 \frac{R_0^4}{\nu^2}\|f\|^2,$$

and consequently,

$$e_0 \leq \frac{8}{\pi^4}\frac{R_0}{\nu^2}\|f\|^2.$$

Therefore on the weak attractor

$$C_0\frac{e_0}{T} = \frac{C_0}{T}e_0^{1/2}e_0^{1/2} \leq \frac{\sqrt{8}C_0}{\pi^2}\frac{1}{T}\frac{R_0^2}{\nu}\frac{\|f\|}{R_0^{3/2}}e_0^{1/2}, \tag{4.5}$$

and so (4.4) yields the allowing.

Theorem 2 *Assume that $u(t) \in \mathcal{A}_w$ and*

$$T \geq \frac{2C_0}{\pi^2}\frac{R_0^2}{\nu}. \tag{4.6}$$

Then

$$\varepsilon_0 \leq 2\sqrt{2}\frac{\|f\|}{R_0^{3/2}} e_0^{1/2}. \tag{4.7}$$

Remark 3 Notice that since $K_2 > 1$, the bound (4.7) is incompatible with the assumption (3.13), which means that Theorem 1 cannot be applied in this particular case if the integral domain is the whole periodic box Ω. Of course, the assumptions of the cascade theorem might still hold on a small enough subregion of Ω. In Sect. 5, we will modify conditions (3.13) and (3.14) to ensure a cascade on the global domain.

4.2 A Lower Bound on Energy

In fluid dynamics, it is natural to assume that the bigger is the driving force, the bigger is the velocity (and the kinetic energy), and hence, the Reynolds number of the flow. In this section, our goal is to obtain the lower bound on the kinetic energy e_0 in periodic setting described above. The idea behind the bounds presented here is similar to Dascaliuc et al. (2009), where the lower bound on energy was obtained in a different context.

In what follows, we assume $f \in H^1$ (or $f \in D(A^{1/2})$).

Then, by duality, multiplying NSE by $\eta A^{-1} f$, and integrating by parts, keeping in mind the divergence-free condition, we obtain:

$$-\int_0^T \eta_t (u, A^{-1} f) + \nu \int_0^T \eta\,(u, f) - \int_0^T \eta\,(B(u, A^{-1} f), u) = \int_0^T \eta\,\|A^{-1/2} f\|^2. \tag{4.8}$$

As before, we bound

$$\left|\frac{1}{T}\frac{1}{R_0^3}\int_0^T \eta_t (u, A^{-1} f)\right| \leq \frac{C_0}{T}\frac{1}{T}\frac{1}{R_0^3}\int_0^T \eta^\delta \|u\|\,\|A^{-1} f\| \leq \frac{\sqrt{2}C_0}{T}\frac{\|A^{-1} f\|}{R_0^{3/2}} e_0^{1/2},$$

$$\left|\frac{1}{T}\frac{1}{R_0^3}\nu\int_0^T \eta\,(f, u)\right| \leq \sqrt{2}\nu\frac{\|f\|}{R_0^{3/2}} e_0^{1/2}$$

and use the Agmon inequality $\|w\|_\infty \leq C_A \|A^{1/2} w\|^{1/2} \|Aw\|^{1/2}$ to bound the trilinear term:

$$\left|\frac{1}{T}\frac{1}{R_0^3}\int_0^T \eta\,(B(u, A^{-1} f), u)\right| \leq \frac{1}{T}\frac{1}{R_0^3}\int_0^T \eta\, C_A \|f\|^{1/2} \|A^{1/2} f\|^{1/2} \|u\|^2$$
$$= 2C_A \|f\|^{1/2} \|A^{1/2} f\|^{1/2}\, e_0.$$

Finally,

$$\frac{1}{T}\frac{1}{R_0^3}\int_0^T \eta\,\|A^{-1/2}f\|^2 = c_\eta \frac{\|A^{-1/2}f\|^2}{R_0^3},$$

where $c_\eta \in (0,1)$ is defined by

$$c_\eta = \frac{1}{T}\int_0^T \eta$$

Then, (4.8) implies:

$$c_\eta \frac{\|A^{-1/2}f\|^2}{R_0^3} \le \sqrt{2}\left(\frac{C_0}{T}\frac{\|A^{-1}f\|}{\|f\|}+\nu\right)\frac{\|f\|}{R_0^{3/2}}e_0^{1/2} + 2C_A\|f\|^{1/2}\|A^{1/2}f\|^{1/2}\,e_0. \tag{4.9}$$

Note that if (4.6) holds, i.e.

$$T \ge \frac{2C_0}{\pi^2}\frac{R_0^2}{\nu},$$

then, using the Poincaré inequality $\|A^{-1}f\| \le (2/\pi)^2\, R_0^2\,\|f\|$,

$$\frac{C_0}{T}\frac{\|A^{-1}f\|}{\|f\|}+\nu \le \frac{\pi^2}{2}\frac{\nu}{R_0^2}\left(\frac{2}{\pi}\right)^2 R_0^2 + \nu = 3\nu.$$

For now, we will denote

$$\gamma_f = \frac{C_0}{\nu T}\frac{\|A^{-1}f\|}{\|f\|}+1,$$

remembering that when (4.6) holds, then $\gamma_f \le 3$. Note however that for any T, by choosing rough enough f, we can make γ_f as close to 1 as needed.

We can view (4.9) as a quadratic inequality in $e_0^{1/2}$. Thus, we solve

$$e_0^{1/2} \ge \frac{-\sqrt{2}\gamma_f\nu\|f\|/R_0^{3/2} + \sqrt{2\gamma_f^2\nu^2\|f\|^2/R_0^3 + 8C_A\,\|f\|^{1/2}\|A^{1/2}f\|^{1/2}c_\eta\|A^{-1/2}f\|^2/R_0^3}}{4C_A\,\|f\|^{1/2}\|A^{1/2}f\|^{1/2}} \tag{4.10}$$

Note that for $g, h > 0$ and $\alpha \in (0,1)$,

$$-h+\sqrt{h^2+g^2} \ge \alpha\, g \quad \Leftrightarrow \quad g \ge \frac{2\alpha}{1-\alpha^2}h.$$

Apply the equivalence above to (4.10) with $\alpha = 1/\sqrt{2}$, $g^2 = 8c_\eta C_A\,\|f\|^{1/2}\|A^{1/2}f\|^{1/2}\|A^{-1/2}f\|^2$ and $h^2 = 2\gamma_f^2\nu^2\|f\|^2$ to obtain

$$e_0^{1/2} \geq \frac{\frac{1}{\sqrt{2}}\, g}{4R_0^{3/2} C_A \,\|f\|^{1/2}\|A^{1/2}f\|^{1/2}} = \frac{1}{2}\frac{c_\eta^{1/2}}{C_A^{1/2}}\frac{1}{R_0^{3/2}}\frac{\|A^{-1/2}f\|}{\|f\|^{1/4}\|A^{1/4}f\|^{1/4}}, \tag{4.11}$$

provided

$$8c_\eta C_A \,\|f\|^{1/2}\|A^{1/2}f\|^{1/2}\|A^{-1/2}f\|^2 \geq (2\sqrt{2})^2 2\gamma_f^2 \nu^2 \|f\|^2 \tag{4.12}$$

We collect the result above in the following theorem.

Theorem 3 *Assume*

$$\|f\| \geq \frac{2\gamma_f^2\nu^2}{c_\eta C_A}\frac{\|f\|^{1/2}}{\|A^{1/2}f\|^{1/2}}\frac{\|f\|^2}{\|A^{-1/2}f\|^2}, \tag{4.13}$$

where

$$\gamma_f = \frac{C_0}{\nu T}\frac{\|A^{-1}f\|}{\|f\|} + 1 \quad \textit{and} \quad c_\eta = \frac{1}{T}\int_0^T \eta \tag{4.14}$$

Then

$$e_0 \geq \frac{c_\eta}{4C_A}\frac{1}{R_0^{5/2}}\frac{\|A^{-1/2}f\|^2}{\|f\|^{3/2}\|A^{1/2}f\|^{1/2}}\frac{\|f\|}{R_0^{1/2}}. \tag{4.15}$$

4.3 One-Sided Kolmogorov's Dissipation Law

Suppose now that the assumptions of both Theorems 2 and 3 hold, i.e. $f \in D(A^{1/2})$, $u \in \mathcal{A}_w$, and (4.6) and (4.13) hold.

Then,

$$\begin{aligned}\varepsilon_0 &\leq \frac{2\sqrt{2}}{R_0^{3/2}} e_0^{1/2}\,\|f\| \leq \frac{2\sqrt{2}}{R_0^{3/2}} e_0^{1/2} R_0^{1/2}\frac{4C_A}{c_\eta} R_0^{5/2}\frac{\|f\|^{3/2}\|A^{1/2}f\|^{1/2}}{\|A^{-1/2}f\|^2} e_0 \\ &= \frac{8\sqrt{2}C_A}{c_\eta} R_0^{5/2}\frac{\|f\|^{3/2}\|A^{1/2}f\|^{1/2}}{\|A^{-1/2}f\|^2}\frac{e_0^{3/2}}{R_0}.\end{aligned}$$

Thus, we have the following result.

Theorem 4 *Assume* $f \in D(A^{1/2})$, $u \in \mathcal{A}_w$,

$$\|f\| \geq \sigma_f \quad \textit{and} \quad T \geq c_1\frac{R_0^2}{\nu}. \tag{4.16}$$

Then:

$$\varepsilon_0 \le 2\sqrt{2}\frac{\|f\|}{R_0^{3/2}}\, e_0^{1/2}, \tag{4.17}$$

$$e_0 \ge \theta_f \frac{\|f\|}{R_0^{1/2}}, \tag{4.18}$$

and

$$\varepsilon_0 \le \frac{2\sqrt{2}}{\theta_f}\frac{e_0^{3/2}}{R_0}, \tag{4.19}$$

where

$$c_1 = \frac{2C_0}{\pi^2}, \quad \sigma_f = \frac{2\gamma_f^2\nu^2}{c_\eta C_A}\frac{\|f\|^{1/2}}{\|A^{1/2}f\|^{1/2}}\frac{\|f\|^2}{\|A^{-1/2}f\|^2}, \quad \theta_f = \frac{c_\eta}{4C_A}\frac{1}{R_0^{5/2}}\frac{\|A^{-1/2}f\|^2}{\|f\|^{3/2}\|A^{1/2}f\|^{1/2}}.$$

As can be seen from Theorem 4, the classical Kolmogorov dissipation law (1.2), an important part of turbulent phenomenology, can be viewed as saturation of the a priori bound (4.19). The non-dimensional shape factors of the force σ_f and θ_f, while invariant under scaling can be small for rough enough $f \in D(A^{1/2})$.

We note that the rigorous one-sided bounds in Kolmogorov's dissipation law were obtained previously in various settings (Constantin and Doering 1994; Foias 1997; Doering and Foias 2002; Dascaliuc et al. 2009).

In the next section, we will investigate several consequences of saturation of this bound.

4.4 Turbulent Scaling

Assume that the solution u, in addition to the conditions of Theorem 4, saturates (4.19), i.e. Kolmogorov's dissipation law holds:

$$\text{There exists } K > 0 \text{ such that } \quad K\frac{e_0^{3/2}}{R_0} \le \varepsilon_0 \le \frac{2\sqrt{2}}{\theta_f}\frac{e_0^{3/2}}{R_0}. \tag{4.20}$$

Then we can prove the following theorem.

Theorem 5 *Assume* (4.16) *and* (4.20) *hold. Then*

$$\frac{K^{3/2}\theta_f^{3/2}}{8^{1/4}}\frac{\|f\|}{R_0^{3/2}}\, e_0^{1/2} \le \varepsilon_0 \le 2\sqrt{2}\frac{\|f\|}{R_0^{3/2}}\, e_0^{1/2}, \tag{4.21}$$

$$\theta_f\frac{\|f\|}{R_0^{1/2}} \le e_0 \le \frac{2\sqrt{2}}{K}\frac{\|f\|}{R_0^{1/2}}, \tag{4.22}$$

and

$$K\theta_f^{3/2}\frac{\|f\|^{3/2}}{R_0^{7/4}} \le \varepsilon_0 \le \frac{8^{5/4}}{K^{3/2}\theta_f}\frac{\|f\|^{3/2}}{R_0^{7/4}}. \tag{4.23}$$

Proof Using (4.17) together with (4.20), we obtain:

$$K\frac{e_0^{3/2}}{R_0} \le 2\sqrt{2}\frac{\|f\|}{R_0^{3/2}}e_0^{1/2},$$

and thus

$$e_0 \le \frac{2\sqrt{2}}{K}\frac{\|f\|}{R_0^{1/2}},$$

so (4.22) holds.

Using (4.22) together with (4.20), we obtain

$$K\frac{\theta_f^{3/2}\frac{\|f\|^{3/2}}{R^{3/4}}}{R_0} \le \varepsilon_0 \le \frac{2\sqrt{2}}{\theta_f}\frac{\left(\frac{2\sqrt{2}}{K}\right)^{3/2}\frac{\|f\|^{3/2}}{R^{3/4}}}{R_0},$$

and (4.23) follows.

Finally, using (4.23) together with (4.22) we obtain:

$$\frac{K^{3/2}\theta_f^{3/2}}{8^{1/4}}\frac{\|f\|}{R_0^{3/2}}e_0^{1/2} \le \frac{K^{3/2}\theta_f^{3/2}}{8^{1/4}}\frac{\|f\|}{R_0^{3/2}}\left(\frac{2\sqrt{2}}{K}\frac{\|f\|}{R_0^{1/2}}\right)^{1/2} \le K\theta_f^{3/2}\frac{\|f\|^{3/2}}{R_0^{7/4}} \le \varepsilon_0,$$

and thus (4.21) holds. □

Remark 4 If $K = c/\theta_f$, with $0 < c < 2\sqrt{2}$, i.e. inequality (4.20) is "fully" saturated, then (4.21)–(4.23) become:

$$\frac{c^{3/2}}{8^{1/4}}\frac{\|f\|}{R_0^{3/2}}e_0^{1/2} \le \varepsilon_0 \le 2\sqrt{2}\frac{\|f\|}{R_0^{3/2}}e_0^{1/2},$$

$$\theta_f\frac{\|f\|}{R_0^{1/2}} \le e_0 \le \frac{2\sqrt{2}}{c}\theta_f\frac{\|f\|}{R_0^{1/2}},$$

and

$$c\theta_f^{1/2}\frac{\|f\|^{3/2}}{R_0^{7/4}} \le \varepsilon_0 \le \frac{8^{5/4}}{c^{3/2}}\theta_f^{1/2}\frac{\|f\|^{3/2}}{R_0^{7/4}}.$$

Remark 5 In fact, proceeding similarly to the proof above, one can show that saturation of any one of the inequalities (4.17)–(4.19) brings about the same conclusions as Theorem 5, with slightly different absolute constants (the parts of the constants depending on the shape of force will be the same.)

Remark 6 Theorem 5 is reminiscent of the main results of Dascaliuc et al. (2009). There the framework of Fourier scales and stationary statistical solutions was used to show that the saturation of Kolmogorov's dissipation law induces similar scalings on energy and its dissipation rate, namely $\sim Gr$ and $\sim Gr^{3/2}$, respectively, where $Gr = \|f\|/\nu^2 R_0^{-3/2}$ is the Grashof number (2.25). In that particular framework, $Gr \gg 1$ would *automatically* imply energy cascade in Fourier scales on the inertial range $[\tau_0, r_f]$, where r_f is the scale of the force. The reason for this is that the aforementioned Gr-scaling implies $\tau_0 (= \kappa_\tau^{-1}) \sim Gr^{-1/4} \ll 1$ (see Dascaliuc et al. 2009), and thus, the sufficient condition for cascades given in Foias et al. (2001b, 2001c) holds.

5 Global Cascades in Periodic Setting

In this section, assume that u is a suitable solution of the space-periodic NSE in Scheffer (1977) and Caffarelli et al. (1982) sense and as before, denote $\Omega = [0, L]^3$—the basic periodic box. Recall (see "Appendix") that u must also be a weak solution satisfying the (localized in time) global energy inequality (2.24), and therefore, the results of Sect. 4 apply.

Our goal here is to develop a sufficient condition for energy cascades, similar to Theorem 1, applicable globally on an integral domain consisting of the entire periodic box Ω.

First, note that in this case we can modify our averages to reflect periodicity:

(1) The cut-off functions will be considered L-periodic: we can extend each ψ_i periodically, provided $\mathrm{Supp}(\psi_i)$ is inside a cubic box of size L (i.e. the radius R of the covering balls should be less then $R_0 := L/4$.)
(2) Otherwise, on each box of size L we will still require a (K_1, K_2)-covering.
(3) The global (space) cut-off therefore can be taken $\psi_0 = 1$ with no special condition on "boundary" elements of the covering.
(4) To accommodate time cut-offs, we will again use the modified energy inequality (2.24).

Thus, the energy and energy dissipation rate—e_0 and ϵ_0 defined in Sect. 4.1 are the global energy and its dissipation rate defined for the integral domain $\Omega_0 = \Omega$.

Recall that in this settings Eq. (4.3) implies that Theorem 1 is impossible on Ω (see Remark 3.)

5.1 A Better Treatment of $\langle (f, u) \rangle_R$ Term

Recall that the first condition of the cascade Theorem 1 was consequence of estimating

$$\langle (f, u) \rangle_R = \frac{1}{R^3} \frac{1}{T} \frac{1}{n} \sum_{i=1}^{n} \iint f \cdot u \, \phi_i \, \mathrm{d}x \mathrm{d}t,$$

where $\phi_i = \eta\psi_i$ are the space–time cut-offs. It turns out that, in the periodic case, we can obtain a better bound for the averages on the global domain Ω.

Lemma 1 *Assume $u \in \mathcal{A}_w$ and the following conditions are satisfied:*

$$\frac{2\sqrt{2}}{K}\frac{\|f\|}{R_0^{3/2}}e_0^{1/2} \le \varepsilon_0. \tag{5.1}$$

and

$$T \ge \frac{4KC_0}{\pi^2}\frac{R_0^2}{\nu}, \tag{5.2}$$

where $K > 1$ is a certain constant that can depend on shape of f.

Then for any collection of the cut-offs $\{\psi_i\}$ associated with a (K_1, K_2)-average, we have

$$\left|\iint (\eta\sum_{i=1}^{n}\psi_i)\, f\cdot u\,\mathrm{d}x\mathrm{d}t\right| \le KK_2\int_0^T \eta\,(f,u)\,\mathrm{d}t. \tag{5.3}$$

(In particular, the integral in the right-hand side must be positive, which is indeed the case—see (5.4) below.)

Proof As in the proof of Theorem 2, we obtain, using the energy balance (4.3),

$$\frac{1}{T}\frac{1}{R_0^3}\int_0^T \eta\,(f,u)\,\mathrm{d}t = \varepsilon_0 - \frac{1}{T}\frac{1}{R_0^3}\int_0^T \eta_t\,\frac{\|u\|^2}{2}\,\mathrm{d}t \ge \varepsilon_0 - \frac{C_0}{T}e_0.$$

Next, apply (4.5) to the last term of the inequality above:

$$\frac{C_0}{T}e_0 \le \frac{2\sqrt{2}C_0}{\pi^2}\frac{1}{T}\frac{R_0^2}{\nu}\frac{\|f\|}{R_0^{3/2}}e_0^{1/2} \le \frac{\sqrt{2}}{K}\frac{\|f\|}{R_0^{3/2}}e_0^{1/2}.$$

Thus, using (5.1) to estimate ε_0, we obtain

$$\frac{1}{T}\frac{1}{R_0^3}\int_0^T \eta\,(f,u)\,\mathrm{d}t \ge \frac{2\sqrt{2}}{K}\frac{\|f\|}{R_0^{3/2}}e_0^{1/2} - \frac{\sqrt{2}}{K}\frac{\|f\|}{R_0^{3/2}}e_0^{1/2} = \frac{\sqrt{2}}{K}\frac{\|f\|}{R_0^{3/2}}e_0^{1/2}. \tag{5.4}$$

Recall that $\sum_i \psi_i \le K_2$, therefore

$$\begin{aligned}\left|\iint_{[0,T]\times\Omega} (\eta\sum_i\psi_i)\, f\cdot u\,\mathrm{d}x\mathrm{d}t\right| &\le \iint_{[0,T]\times\Omega} K_2\eta\,|f||u|\,\mathrm{d}x\mathrm{d}t \le K_2T^{\frac{1}{2}}\,\|f\|\left(\int_0^T \|u\|^2\,\eta^\delta\,\mathrm{d}t\right)^{\frac{1}{2}}\\ &= TR_0^3\,K_2\sqrt{2}\frac{\|f\|}{R_0^{3/2}}e_0^{1/2}.\end{aligned} \tag{5.5}$$

Now use (5.4) to obtain

$$\left| \iint\limits_{[0,T]\times\Omega} \left(\eta \sum_i \psi_i \right) f \cdot u \,\mathrm{d}x\mathrm{d}t \right| \leq K_2 K \left(T R_0^3 \frac{\sqrt{2}}{K} \frac{\|f\|}{R_0^{3/2}} e_0^{1/2} \right) = K K_2 \int_0^T \eta\,(f,u)\,\mathrm{d}t.$$

□

We are now ready to estimate the $\langle (f,u) \rangle_R$-term.

Lemma 2 *Under the assumptions on* Lemma 1, *for any* (K_1, K_2)*-average on scale* R *we have:*

$$|\langle (f,u)\rangle_R| = \left| \frac{1}{T} \frac{1}{R^3} \frac{1}{n} \sum_{i=1}^n \iint (\eta\psi_i)\, f \cdot u \,\mathrm{d}x\mathrm{d}t \right| \leq K K_2 \frac{\tau_{-1}}{\tau_f} \frac{R_0^3}{R^3} \frac{1}{n} \frac{\|f\|}{R_0^{3/2}} e_0^{1/2}, \tag{5.6}$$

where

$$\tau_f = \frac{\|f\|}{\|A^{\frac{1}{2}} f\|} \quad \textit{and} \quad \tau_{-1} = \left(\frac{\frac{1}{T}\frac{1}{R_0^3} \int_0^T \eta^2 \,\|A^{-\frac{1}{2}} u\|^2 \,\mathrm{d}t}{e_0} \right)^{\frac{1}{2}} \tag{5.7}$$

Proof We start by noticing that

$$\begin{aligned} \left| \int_0^T \eta\,(f,u)\,\mathrm{d}t \right| &\leq \left| \int_0^T \eta\,(A^{1/2} f, A^{-1/2} u)\,\mathrm{d}t \right| \leq \int_0^T \eta\,\|A^{1/2} f\|\,\|A^{-1/2} u\|\,\mathrm{d}t \\ &\leq T^{\frac{1}{2}} \|A^{\frac{1}{2}} f\| \left(\int_0^T \eta^2 \,\|A^{-\frac{1}{2}} u\|^2 \,\mathrm{d}t \right)^{\frac{1}{2}} = T^{\frac{1}{2}} \frac{\tau_{-1}}{\tau_f} \|f\| \left(\int_0^T \eta^\delta \frac{\|u\|^2}{2} \,\mathrm{d}t \right)^{\frac{1}{2}}. \end{aligned} \tag{5.8}$$

Now, by Lemma 1,

$$|\langle (f,u)\rangle_R| = \left| \frac{1}{T} \frac{1}{R^3} \frac{1}{n} \iint \left(\eta \sum_{i=1}^n \psi_i \right) f \cdot u \,\mathrm{d}x\mathrm{d}t \right| \leq \frac{1}{T} \frac{1}{R^3} \frac{1}{n} K K_2 \int_0^T \eta\,(f,u)\,\mathrm{d}t,$$

which, together with (5.8) implies (5.6). □

Since τ_{-1} is the crucial scale in previous lemma, it is interesting to compare it to the Taylor scale τ_0.

Lemma 3 *Assume* $1/2 < \delta < 1$. *Let*

$$\tilde{\tau}_{-1} = \left(\frac{\frac{1}{R_0^3} \frac{1}{T} \int_0^T \eta^{2\delta - 1} \|A^{-1/2} u\|^2}{e_0} \right)^{\frac{1}{2}} \tag{5.9}$$

then

$$\tilde{\tau}_{-1} \geq 2\, \tau_0. \tag{5.10}$$

Proof Note that we have:

$$\begin{aligned} 2T R_0^3 e_0 &= \int_0^T \eta^{\delta} \|u\|^2 = \int_0^T (\eta^{\frac{1}{2}} A^{\frac{1}{2}} u, \eta^{\delta - \frac{1}{2}} A^{-\frac{1}{2}} u)\, \mathrm{d}x \mathrm{d}t \\ &\leq \left(\int_0^T \eta \|A^{\frac{1}{2}} u\|^2 \right)^{\frac{1}{2}} \left(\int_0^T \eta^{2\delta - 1} \|A^{-\frac{1}{2}} u\|^2 \right)^{\frac{1}{2}} \end{aligned}$$

and thus, (5.10) follows (if we recall that $\tau_0 = (\nu e_0/\varepsilon_0)^{1/2}$, and ε_0 is given by (4.2)). □

Remark 7 Notice that $\tau_{-1} \leq \tilde{\tau}_{-1}$. Thus, Lemma 2 holds with τ_{-1} replaced by $\tilde{\tau}_{-1}$. (Arguably, both τ_{-1} and $\tilde{\tau}_{-1}$ should in fact be of comparable size.) Also, (5.4) and (5.5) imply that the terms $\int \eta\,(f, u)\,\mathrm{d}t$ and $\|f\| e_0^{1/2}$ have *the same scaling*: $\sim \mathrm{Gr}^{3/2}$. In particular, this means that *the force, f, and the solution, u, are aligned in H over* $[0, T]$, since *the Cauchy–Schwarz inequality for* (f, u) *is saturated*. Therefore, the estimate (5.6) is sharp (as an order of Gr), and, in order to control the force-related term in the energy balance, we will need to require that "the bulk of oscillations" of f (characterized by the scale τ_f) does not extend below the Taylor scale τ_0.

Remark 8 Notice that qualitatively both (5.6) and (3.6) look similar; however, the *size* of the constant in front of $\|f\| e_0^{1/2}$ (which led to the constant α_0 in the assumption (3.13) in Theorem 1—see Remark 3) is crucial in establishing the cascade inside the entire periodic box. In this sense, the bound (5.6) becomes better, provided the quotient τ_{-1}/τ_f is small enough.

5.2 A Better Cascade Theorem on Ω

Using the improved bound (5.6) instead of (3.6), we obtain the following theorem.

Theorem 6 *Let* $C \in (0, 2^{3/2})$ *and* $\alpha_0 \in (0, 1)$ *be fixed. Assume* $f \in D(A^{1/2})$ *and* $T > 0$ *be such that*

$$\|f\| \geq 2^3 \frac{C_0^2 K_2^2}{C^4 \alpha_0^2}\, \theta_f\, \frac{\nu^2}{R_0^{3/2}} \tag{5.11}$$

and

$$T \geq \frac{2^{13/4} C_0}{\pi^2 C^{3/2}} \frac{R_0^2}{\nu}. \tag{5.12}$$

If $u \in \mathcal{A}_w$ is a suitable and Leray–Hopf solution that saturates Kolmogorov's dissipation law on the time interval $[0, T]$*, i.e.*

$$\frac{C}{\theta_f} \frac{e_0^{3/2}}{R_0} \leq \varepsilon_0 \left(\leq \frac{2\sqrt{2}}{\theta_f} \frac{e_0^{3/2}}{R_0} \right), \tag{5.13}$$

and

$$\frac{2^{11/4} K_2}{\alpha_0 C^{3/2}} \tau_{-1} \leq \tau_f, \tag{5.14}$$

then we will have energy cascade

$$\frac{(1-\alpha_0)}{K_1} \varepsilon_0 \leq \langle \Phi \rangle_R \leq K_2 (1+\alpha_0) \varepsilon_0 \tag{5.15}$$

for any (K_1, K_2)*-average on* Ω *for all* R *inside the inertial range*

$$[\beta_0 \tau_0, \ R_0], \quad \textit{where} \quad \beta_0 := \left(\frac{2}{\alpha_0} C_0 K_2 \right)^{1/2}. \tag{5.16}$$

Moreover, in this case

$$\frac{C^{3/2}}{2^{15/4}} \theta_f^{1/2} \frac{\nu R_0^{5/4}}{\|f\|^{1/2}} \leq \tau_0^2 \leq \frac{2^{3/2}}{C^2} \theta_f^{1/2} \frac{\nu R_0^{5/4}}{\|f\|^{1/2}} \tag{5.17}$$

Proof First, note that by Theorem 5, the assumptions (5.13) and (5.12) imply (5.1) and (5.2) with

$$K = \frac{1}{2^{3/4} C^{3/2}}.$$

Therefore, the estimate from Lemma 2 holds.

We proceed as in the proof of Theorem 1, but use (5.6) instead of (3.6) to arrive to the following version of (3.7):

$$\begin{aligned} \frac{1}{n} \left(\frac{R_0}{R} \right)^3 \left[\varepsilon_0 - C_0 K_2 \frac{\nu}{R^2} e_0 - 2K K_2 \frac{\tau_{-1}}{\tau_f} \frac{\|f\|^2}{R_0^{3/2}} e_0^{1/2} \right] &\leq \langle \Phi \rangle_R \\ \leq K_2 \frac{1}{n} \left(\frac{R_0}{R} \right)^3 \left[\varepsilon_0 + C_0 \frac{\nu}{R^2} e_0 + 2K \frac{\tau_{-1}}{\tau_f} \frac{\|f\|^2}{R_0^{3/2}} e_0^{1/2} \right]&, \end{aligned} \tag{5.18}$$

Now use (5.1) to bound $\frac{\|f\|^2}{R_0^{3/2}}\, e_0^{1/2}$, then apply (3.8) and factor out ε_0:

$$\frac{1}{K_1}\,\varepsilon_0\left(1-C_0K_2\frac{\tau_0^2}{R^2}-2^{1/2}K^2K_2\frac{\tau_{-1}}{\tau_f}\right)\leq\langle\Phi\rangle_R\leq K_2\varepsilon_0\left(1-C_0\frac{\tau_0^2}{R^2}-2^{1/2}K^2\frac{\tau_{-1}}{\tau_f}\right)$$

Notice that if

$$C_0K_2\frac{\tau_0^2}{R_0^2}\leq\frac{\alpha_0}{2} \tag{5.19}$$

and

$$2^{1/2}K^2K_2\frac{\tau_{-1}}{\tau_f}\leq\frac{\alpha_0}{2}, \tag{5.20}$$

then (5.15) holds for R inside the interval given in (5.16).

Obviously, (5.20) is equivalent to (5.14), as for (5.19), it indeed holds if the upper bound of (5.17) and (5.11) hold.

It remains to prove (5.17) holds. But it immediately follows from the bounds on e_0 and ε_0 from Remark 4. □

The main advantage of the preceding result is that it establishes conditions for the energy cascade inside the *entire* periodic box Ω, where Theorem 1 fails (see Remark 3). An additional benefit of Theorem 6 is the sharp estimate on the width of the Taylor scale (and thus the width of the inertial range) in terms of the magnitude of the force (5.17).

Note that conditions (5.11) and (5.12) are rather straightforward and consistent with turbulence phenomenology, which requires high Reynolds numbers and long-time averages. Assumption (5.13) is a version of the dissipation law (1.2) and therefore again consistent with the empirical theory. Finally, as was mentioned in Remark 7, the condition (5.14) is in fact a rather mild one. It states that the "force scale" τ_f is not too small, i.e. f acts mostly on the big scale. Thus, essentially, the above theorem establishes that Kolmogorov's dissipation law will trigger energy cascade in physical scales inside a periodic domain, provided the force is mild enough (i.e. the force scale τ_f dominates the scale τ_{-1}). Theorem 5 (more precisely Remark 4) implies that, in terms of Grashof number (2.25), the width of the inertial range (in reciprocals of the length scales) is of order $Gr^{1/4}$, and thus, the cascades are wider for higher magnitude forces. Finally, by the same theorem, the Reynolds number, $Re = e_0^{1/2}/(\nu R_0^{-1})$, grows like $Gr^{1/2}$, and so our results prove that Kolmogorov's dissipation law and high Reynolds numbers imply wide energy cascades in the physical scales of periodic flows.

We conclude with several remarks.

Remark 9 Further comparing Theorems 1 with 6, we notice that in the periodic case, saturation of the Kolmogorov dissipation law (5.13) is equivalent to the saturation of the a priori bound (4.7) (see Remark 5.) Therefore (5.13) corresponds to the condition (3.13) in Theorem 1. Thus, philosophically, Theorem 1 states that in generic settings, saturation of Kolmogorov dissipation law plus small Taylor scale imply energy cascade. Theorem 6 states that in the case of the whole domain in periodic case, if force is big enough, Kolmogorov law in fact *implies* that Taylor scale is small and is alone responsible for the cascade.

Remark 10 Overall, the results of Sect. 5, obtained for the physical scales, mirror closely the ones obtained in Dascaliuc et al. (2009) in the framework of Fourier scales and statistical solutions (see Remark 6). Recall, Dascaliuc et al. (2009) shows that Kolmogorov's dissipation law (defined there in terms of generalized limits) induces *the same* restrictive scaling on energy and its dissipation rate as the one obtained here. Recall that, as noted in Remark 6, in the context of Fourier scales, as $Gr \to \infty$, this scaling combined with the cascade condition from Foias et al. (2001b), *automatically* implies energy cascade up to the Taylor scale.

Remark 11 We should stress that Kolmogorov's dissipation law (5.13) is responsible for the peculiar alignment of f and u, which makes the estimate (5.6) possible. In general case, in the absence of additional information on alignment of f and u, it is impossible to a priori eliminate the case when the Cauchy–Schwarz inequality $(f, u) \leq \|f\|\,\|u\|$ saturated on the interval $[0, T]$ even if one assumes $\tau_0 \ll \|f\|/\|A^{1/2}f\|$ (which would mean that f is geometrically very different from u.)

For example, one can construct vectors in H: $u = v_1 + \alpha v_n$ and $f = v_1$, where $\{v_i\}$ is an orthonormal basis of eigenvectors of A, arranged in the increasing order of eigenvalues. If $\alpha = \lambda_n^{-1/4}$ (λ_n is an eigenvalue corresponding to v_n) an $n \gg 1$, then we will in fact have $(f, u) \approx \|f\|\,\|u\|$, while "$\tau_0$"$= \|u\|/\|A^{1/2}u\| \ll \|f\|/\|A^{1/2}f\|$. If something similar happens for an actual solution of the NSE on $[0, T]$, then (3.6) is the best estimate available, which, as we noticed in Remark 3, will not be useful in establishing cascade on the whole Ω.

Remark 12 Notice that because of periodicity, the global flux on Ω is zero:

$$\Phi_\Omega = \frac{1}{T}\frac{1}{R_0^3}\int_0^T \eta\,(B(u, u), u)\,\mathrm{d}t = 0.$$

Therefore, the flux density $(u \cdot \nabla u)u \cdot u + \nabla p$ *must be a sign-varying quantity*. The fact that *all* the (K_1, K_2)-averages across a wide range of scales are comparable to ε_0 is therefore truly significant.

6 Appendix: The Link Between Suitable and Leray–Hopf Solutions in the Periodic Case

As mentioned above, in the space-periodic settings the suitable weak solutions of the NSE will in fact also be the Leray–Hopf solutions with a slightly modified global energy inequality (the inequality is localized in time, in contrast to the classical Leray–Hopf case). For completeness, we present the poof of this fact. It is worth noticing that the converse statement is an open problem, linked with the general regularity problem of the NSE.

Theorem 7 *Let u, p be a suitable solution on $[0, T]$ to the space-periodic NSE. Then u will satisfy the NSE in V' (i.e. u is a weak solution—see (2.22)), and for any time cut-off $\eta \in C_0^1([0, T])$, the global energy inequality will hold:*

$$\nu \int_0^T \eta \, \|\nabla u\|^2 \leq \int_0^T \eta_t \frac{\|u\|^2}{2} + \int_0^T \eta \, (f, u). \tag{6.1}$$

Proof Let $L > 0$ be the period, and $\Omega = [0, L]^3$ be a fixed box of size L. We start with the following claim. □

Claim 1 *There exists a finite family of functions $\{\psi_j\}$ such that:*

(1) *Each $\psi_j \in C^\infty(\mathbb{R}^n)$, ψ_j are L-periodic, $1 \geq \psi_j \geq 0$.*
(2) *For each ψ_j, there exists a size L box $\Omega_j \subset \mathbb{R}^n$, such that the restriction of ψ_j on the* interior *of Ω_j has a compact support.*
(3) *$\sum \psi_j = 1$ on $\mathbb{R}^n$.*

Proof (of the Claim) Let $f_1 \in C^\infty(\mathbb{R})$ be an L-periodic function, $0 \leq f_1 \leq 1$, such that $f_1 = 1$ on $[(1/3)L, (2/3)L]$ and $f_1 = 0$ on $[0, (1/6)L] \cup [(5/6)L, 1]$. Now $f_2 = 1 - f_1$ will also be a C^∞, non-negative, L-periodic function restriction of which on the interior of (the one-dimensional periodic box) $[-L/2, L/2]$ is compactly supported. Moreover, $f_1 + f_2 = 1$ on $\mathbb{R}$.

In $\mathbb{R}^n$ we have

$$1 = \prod_{k=1}^{n} (f_1(x_k) + f_2(x_k)) = \sum_{i_1,\ldots,i_n=1}^{2} f_{i_1}(x_1) \cdot \ldots \cdot f_{i_n}(x_n) = \sum_{i_1,\ldots,i_n=1}^{2} \psi_{i_1,\ldots,i_n}(x),$$

where $\psi_{i_1,\ldots,i_n}(x_1, \ldots, x_n) = f_{i_1}(x_1) \cdot \ldots \cdot f_{i_n}(x_n)$ satisfy the conditions (1)–(3) (after a re-indexing). □

Returning to the proof of the theorem, let ψ_j be the test functions satisfying (1)–(3) in Claim 1. Then on for all j, the local energy inequalities are satisfied:

$$\begin{aligned} \int_0^T \int_{\Omega_j} \left(\frac{|u|^2}{2} + p \right) u \cdot \nabla \phi_j \; &\geq \; \nu \int_0^T \int_{\Omega_j} |\nabla u|^2 \phi_j \\ &- \int_0^T \int_{\Omega_j} \frac{|u|^2}{2} (\partial_t \phi_j - \nu \Delta \phi_j) - \int_0^T \int_{\Omega_j} (f, u) \phi_j, \end{aligned} \tag{6.2}$$

where $\phi_j = \eta \psi_j$ and Ω_j from the Claim 1, (2). But since all the functions involved are L-periodic in x, the integrals over Ω_j equal to the integrals over Ω a.e. in t. Thus, writing Ω instead of Ω_j in (6.2) and summing up in j (keeping in mind $\sum \phi_j = \sum \eta \psi_j = \eta$), we obtain (6.1).

It remains to show u solves the NSE in V'. But since u is a suitable weak solution, then $u \in L^2([0, T], V)$ and solves the NSE in the sense of distributions:

$$-\int_0^T\int_{\mathbb{R}^3} u\cdot\phi_t - \int_0^T\int_{\mathbb{R}^3}(\nabla u:\nabla\phi) - \int_0^T\int_{\mathbb{R}^3}(u\cdot\nabla)\phi\cdot u = \int_0^T\int_{\mathbb{R}^3} f\cdot\phi\,, \tag{6.3}$$

for any divergence-free $\phi\in C_0^\infty(\mathbb{R}\times\mathbb{R}^3,\mathbb{R}^3)$. Note that since u, p, and f are space periodic, then by extending C_0^∞ test functions with the support inside an L-periodic box to the whole $\mathbb{R}^3$ (and considering sums of such functions, like in the proof of Claim 1), we can see that (6.3) also holds for ϕ that are C^∞ and L-periodic in the space variable, integrated over Ω.

But the density properties of C^∞ in H^1 imply that for any $v\in V$ there exists a sequence of divergence-free space periodic C^∞-functions, $\{\psi_k\}$ such that $\psi_k\to v$ in $H^1(\Omega,\mathbb{R}^3)$. Then for $\phi_k=\eta\psi_k$, distributional pairing above implies, as $k\to\infty$, the duality pairing of the NSE with v in V':

$$-\int_0^T\int_\Omega \eta_t\, u\cdot v - \int_0^T\int_\Omega \eta\,(\nabla u:\nabla v) - \int_0^T\int_\Omega \eta\,(u\cdot\nabla)v\cdot u = \int_0^T\int_\Omega \eta\, f\cdot v\,. \tag{6.4}$$

This means that for any $v\in V$, in the sense of distributions,

$$F_t = G,$$

where, using the notation from Sect. 2.3, $F=(u,v)$, $G=(B(u,v),u)-(A^{1/2}u,A^{1/2}v)+(f,v)$. Thus, (2.22) is satisfied, i.e. the suitable weak solution u satisfies the NSE in V'. □

Remark 13 Notice that by (6.4), both F and G are locally L^1 in time (F, G are form the proof of Theorem 7.) Therefore, F is absolutely continuous and $F_t=G$ a.e., in other words, (2.22) *holds a.e. in* t.

The next result is an elementary observation on adapting the Leray–Hopf energy inequality to the localized in time case.

Theorem 8 *Assume u is the Leray–Hopf solution of the space-periodic NSE, and let $\eta\in C^2([t_0,t_1],\mathbb{R}_+)$, where t_0 is the Lebesgue point of $\|u(\cdot)\|^2$, and $t_1>t_0$. Then,*

$$\eta(t_1)\frac{\|u(t_1)\|^2}{2} - \eta(t_0)\frac{\|u(t_0)\|^2}{2} - \int_{t_0}^{t_1}\eta_t\frac{\|u\|^2}{2} \le -\nu\int_{t_0}^{t_1}\eta\,\|\nabla u\|^2 + \int_{t_0}^{t_1}\eta\,(f,u). \tag{6.5}$$

Proof Recall that the Leray–Hopf solutions satisfy the energy inequality (2.23) for all $t_0\ge 0$—Lebesgue points of $\|u(\cdot)\|^2$ and for all $t\ge t_0$. Moreover, the set of Lebesgue points is dense in $[0,\infty)$. Additionally, $u(t)$ is regular on a countable union of open intervals of the full measure inside any $[0,T]$, which means $\|u(t)\|^2$ is continuous a.e. Finally, $\|u(t)\|^2$ is bounded on $[0,\infty)$—a direct consequence of the Leray–Hopf energy inequality (2.23). Therefore, $\|u(t)\|^2$ is Riemann integrable on any compact interval in $[0,\infty)$.

Now let $t_0 \geq 0$ be a Lebesgue point for $\|u(t)\|^2$, $t_1 > t_0$, and let $\eta \in C^2([t_0, t_1], \mathbb{R}_+)$. For a $\delta > 0$, consider a Riemann partition $\{s_j\}_{j=0,n_\delta}$, where each s_j is a Lebesgue point for $\|u(t)\|^2$, $t_0 = s_0 < s_1 < \cdots < s_{n_\delta} = t_1$, and each $s_j - s_{j-1} < \delta$. If for $j < n_\delta$, we define the step functions

$$\eta_j(t) = \begin{cases} \eta(s_j), & t \in [s_j, s_{j+1}); \\ 0, & \text{otherwise}, \end{cases}$$

then, as $\delta \to 0$, $\sum \eta_j$ converges to η a.e. on $[t_0, t_1]$. Also, for each $j = 0, \ldots, n_\delta - 1$, the Leray–Hopf energy inequality (2.23) on $[s_j, s_{j+1}]$ implies:

$$\eta(s_j)\frac{\|u(s_{j+1})\|^2}{2} - \eta(s_j)\frac{\|u(s_j)\|^2}{2} \leq -\nu \int_{t_0}^{t_1} \eta_j \, \|\nabla u\|^2 + \int_{t_0}^{t_1} \eta_j \, (f, u).$$

Summing in j, we obtain

$$\sum_{j=0}^{n_\delta-1} \eta(s_j)\frac{\|u(s_{j+1})\|^2}{2} - \sum_{j=0}^{n_\delta-1} \eta(s_j)\frac{\|u(s_j)\|^2}{2} \leq -\nu \int_{t_0}^{t_1} \sum_{j=0}^{n_\delta-1} \eta_j \, \|\nabla u\|^2 + \int_{t_0}^{t_1} \sum_{j=0}^{n_\delta-1} \eta_j \, (f, u). \tag{6.6}$$

Notice that by the Lebesgue dominated convergence theorem, as $\delta \to 0$, $\int_{t_0}^{t_1} \sum \eta_j \, \|\nabla u\|^2 \to \int_{t_0}^{t_1} \eta \, \|\nabla u\|^2$, while $\int_{t_0}^{t_1} \sum \eta_j \, (f, u) \to \int_{t_0}^{t_1} \sum \eta \, (f, u)$.

To deal with the right-hand side of (6.6), write

$$\begin{aligned} &\sum_{j=0}^{n_\delta-1} \eta(s_j)\frac{\|u(s_{j+1})\|^2}{2} - \sum_{j=0}^{n_\delta-1} \eta(s_j)\frac{\|u(s_j)\|^2}{2} \\ &\quad = \eta(s_{n_\delta-1})\frac{\|u(s_{n_\delta})\|^2}{2} - \eta(s_0)\frac{\|u(s_0)\|^2}{2} - \sum_{j=1}^{n_\delta-1} (\eta(s_j) - \eta(s_{j-1}))\frac{\|u(s_j)\|^2}{2}. \end{aligned}$$

Notice that

$$\begin{aligned} \sum_{j=1}^{n_\delta-1} (\eta(s_j) - \eta(s_{j-1}))\frac{\|u(s_j)\|^2}{2} &= \sum_{j=1}^{n_\delta-1} \eta'(s_j)\frac{\|u(s_j)\|^2}{2}(s_j - s_{j-1}) \\ &\quad - \sum_{j=1}^{n_\delta-1} \eta''(\xi_j)\frac{\|u(s_j)\|^2}{2}(s_j - s_{j-1})^2, \end{aligned}$$

where $\xi_j \in [s_{j-1}, s_j]$. We can identify the first sum in the right-hand side as the Riemann sum (with a missing 0th term) corresponding to $\int_{t_0}^{t_1} \eta_t \frac{\|u\|^2}{2}$, while the absolute value is the second sum is bounded above by:

$$\left| \sum_{j=1}^{n_\delta - 1} \eta''(\xi_j) \frac{\|u(s_j)\|^2}{2} (s_j - s_{j-1})^2 \right| \leq M \delta R_{n_\delta},$$

where $M = \sup\{\eta''(t) : t \in [t_0, t_1]\} < \infty$, and R_{n_δ} is the Riemann sum corresponding to $\int_{t_0}^{t_1} \frac{\|u\|^2}{2} < \infty$. Thus, this term converges to zero as $\delta \to 0$.

Consequently, (6.5) is obtained form (6.6) by letting $\delta \to 0$. □

Remark 14 Clearly, (6.5) implies the usual Leray–Hopf inequality (2.23). Also, (6.5) implies (6.1) for $\eta \in C_0^2((0, T), \mathbb{R}_+)$, and therefore, any Leray–Hopf solution will automatically satisfy the localized in time global energy inequality considered in Sect. 4.

References

Balci, N., Foias, C., Jolly, M.S., Rosa, R.: On universal relations in 2-D turbulence. Discrete Contin. Dyn. Syst. **27**(4), 1327–1351 (2010)

Caffarelli, L., Kohn, R., Nirenberg, L.: Partial regularity of suitable weak solutions of the Navier–Stokes equations. Commun. Pure Appl. Math. **35**(6), 771–831 (1982)

Cheskidov, A.: Blow-up in finite time for the dyadic model of the Navier–Stokes equations. Trans. Am. Math. Soc. **360**(10), 5101–5120 (2008)

Cheskidov, A.: Global attractors of evolutionary systems. J. Dyn. Differ. Equ. **21**(2), 249–268 (2009)

Cheskidov, A., Constantin, P., Friedlander, S., Shvydkoy, R.: Energy conservation and Onsager's conjecture for the Euler equations. Nonlinearity **21**(6), 1233–1252 (2008)

Cheskidov, A., Friedlander, S.: The vanishing viscosity limit for a dyadic model. Phys. D **238**(8), 783–787 (2009)

Cheskidov, A., Shvydkoy, R.: Euler equations and turbulence: analytical approach to intermittency. SIAM J. Math. Anal. **46**(1), 353–374 (2014)

Constantin, P.: The Littlewood–Paley spectrum in two-dimensional turbulence. Theor. Comput. Fluid Dyn. **9**(3–4), 183–189 (1997)

Constantin, P., Doering, C.R.: Variational bounds on energy dissipation in incompressible flows: shear flow. Phys. Rev. E (3) **49**(5, part A), 4087–4099 (1994)

Dascaliuc, R., Foias, C., Jolly, M.S.: Relations between energy and enstrophy on the global attractor of the 2-D Navier–Stokes equations. J. Dyn. Differ. Equ. **17**(4), 643–736 (2005)

Dascaliuc, R., Foias, C., Jolly, M.S.: Universal bounds on the attractor of the Navier–Stokes equation in the energy, enstrophy plane. J. Math. Phys. **48**(6), 065201, 33 (2007)

Dascaliuc, R., Foias, C., Jolly, M.S.: On the asymptotic behavior of average energy and enstrophy in 3D turbulent flows. Phys. D **238**(7), 725–736 (2009)

Dascaliuc, R., Grujić, Z.: Energy cascades and flux locality in physical scales of the 3D Navier–Stokes equations. Commun. Math. Phys. **305**(1), 199–220 (2011)

Dascaliuc, R., Grujić, Z.: Anomalous dissipation and energy cascade in 3D inviscid flows. Commun. Math. Phys. **309**(3), 757–770 (2012a)

Dascaliuc, R., Grujić, Z.: Dissipation anomaly and energy cascade in 3D incompressible flows. C. R. Math. Acad. Sci. Paris **350**(3–4), 199–202 (2012b)

Dascaliuc, R., Grujić, Z.: Vortex stretching and criticality for the three-dimensional Navier–Stokes equations. J. Math. Phys. **53**(11), 115613, 9 (2012c)

Dascaliuc, R., Grujić, Z.: Coherent vortex structures and 3D enstrophy cascade. Commun. Math. Phys. **317**(2), 547–561 (2013)

Doering, C.R., Foias, C.: Energy dissipation in body-forced turbulence. J. Fluid Mech. **467**, 289–306 (2002)

Eyink, G.L.: Locality of turbulent cascades. Phys. D **207**(1), 91–116 (2005)

Foias, C.: What do the Navier–Stokes equations tell us about turbulence? In: Harmonic Analysis and Nonlinear Differential Equations (Riverside, CA, 1995), volume 208 of Contemp. Math., pp. 151–180. Amer. Math. Soc., Providence (1997)

Foias, C., Manley, O., Rosa, R., Temam, R.: Navier–Stokes Equations and Turbulence, Volume 83 of Encyclopedia of Mathematics and Its Applications. Cambridge University Press, Cambridge (2001a)

Foias, C., Manley, O.P., Rosa, R., Temam, R.: Cascade of energy in turbulent flows. C. R. Acad. Sci. Paris Sér. I Math. **332**(6), 509–514 (2001b)

Foias, C., Manley, O.P., Rosa, R., Temam, R.: Estimates for the energy cascade in three-dimensional turbulent flows. C. R. Acad. Sci. Paris Sér. I Math. **333**(5), 499–504 (2001c)

Foias, C., Rosa, R., Temam, R.: Topological properties of the weak global attractor of the three-dimensional Navier–Stokes equations. Discrete Contin. Dyn. Syst. **27**(4), 1611–1631 (2010)

Foias, C., Temam, R.: The connection between the Navier–Stokes equations, dynamical systems, and turbulence theory. In: Directions in Partial Differential Equations (Madison, WI, 1985), volume 54 of Publ. Math. Res. Center Univ. Wisconsin, pp. 55–73. Academic Press, Boston (1987)

Frisch, U.: Turbulence. Cambridge University Press, Cambridge (1995). (The legacy of A. N. Kolmogorov)

Goldstein, R.: Fluid Mechanics Measurements. CRC Press, Boca Raton (1996)

Katz, N., Pavlović, N.: Finite time blow-up for a dyadic model of the Euler equations. Trans. Am. Math. Soc. **357**(2), 695–708 (2005)

Kolmogoroff, A.N.: On degeneration of isotropic turbulence in an incompressible viscous liquid. C. R. (Doklady) Acad. Sci. URSS (N. S.) **31**, 538–540 (1941)

Kolmogorov, A.N.: The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers. Proc. R. Soc. Lond. Ser. A **434**(1890), 9–13 (1991). (Translated from the Russian by V. Levin, Turbulence and stochastic processes: Kolmogorov's ideas 50 years on)

Lemarié-Rieusset, P.G.: Recent Developments in the Navier–Stokes Problem, Volume 431 of Chapman & Hall/CRC Research Notes in Mathematics. Chapman & Hall/CRC, Boca Raton (2002)

Leray, J.: Sur le mouvement d'un liquide visqueux emplissant l'espace. Acta Math. **63**(1), 193–248 (1934)

Otto, F., Ramos, F.: Universal bounds for the Littlewood–Paley first-order moments of the 3D Navier–Stokes equations. Commun. Math. Phys. **300**(2), 301–315 (2010)

Richardson, L.F.: Atmospheric diffusion shown on a distance-neighbour graph. Proc. R. Soc. Lond. A **110**, 709–737 (1926)

Scheffer, V.: Hausdorff measure and the Navier–Stokes equations. Commun. Math. Phys. **55**(2), 97–112 (1977)

Sulem, P.L., Frisch, U.: Bounds on energy flux for finite energy turbulence. J. Fluid Mech. **72**(03), 417–423 (1975)

Tao, T.: Finite time blowup for an averaged three-dimensional Navier–Stokes equation (2014). arXiv:1402.0290

Taylor, G.I.: Statistical theory of turbulence. Proc. R. Soc. Lond. A **A151**, 421–478 (1935)

Temam, R.: Navier–Stokes Equations: Theory and Numerical Analysis, 3rd edn. AMS, Chelsea (1984)

Temam, R.: Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Volume 68 of Applied Mathematical Sciences, 2nd edn. Springer, New York (1997)

Višik, M.I., Fursikov, A.V.: Matematicheskie zadachi statisticheskoi gidromekhaniki. Nauka, Moscow (1980)

Zhang, Q.S.: A strong regularity result for parabolic equations. Commun. Math. Phys. **244**(2), 245–260 (2004)