Electronic Journal of Differential Equations, Vol. **1999**(1999) No. 48, pp. 1–8.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu ftp ejde.math.unt.edu (login: ftp)

ON THE SMALLNESS OF THE (POSSIBLE) SINGULAR SET IN SPACE FOR 3D NAVIER-STOKES EQUATIONS

Zoran Grujić

Abstract

We utilize L^∞ estimates on the complexified solutions of 3D Navier-Stokes equations via a plurisubharmonic measure type maximum principle to give a short proof of the fact that the Hausdorff dimension of the (possible) singular set in space is less or equal 1 assuming chaotic, Cantor set-like structure of the blow-up profile.

1. Introduction

The problem of global regularity for 3D Navier-Stokes equation (NSE) is one of the most challenging problems in the mathematical theory of fluid dynamics. Since the fundamental work of Leray [L] in 1930's, we know the existence of global weak solutions; however, the existence of strong (regular) solutions is known only locally in time. Some partial regularity results appeared already in the Leray's work - later on, various partial regularity results were obtained in [FT1], [Sch1], [Sch2], [CKN] and [St]. The best result up to date is in [CKN] - it implies that for every $T > 0$, one-dimensional Hausdorff measure of the singular set in $\Omega \times (0, T)$ is 0. If we are looking at a snapshot, i.e. at the singular set in space for some fixed singular time T_s, we do not have a better estimate. The best we can say is again that one-dimensional Hausdorff measure of the singular set S_{T_s} in $\Omega \times \{T_s\}$ is 0. The proof of the [CKN] result is based on a local theory - local blow-up estimates on families of shrinking space-time cylinders. The main tools in the proof are localized energy inequality, local interpolation and localized estimates on the pressure. A simplified proof using essentially the same tools appeared recently in [Li].

In this paper, we present a completely different, short proof of the fact that $d_H(S_{T_s}) \leq 1$ (d_H denotes the Hausdorff dimension of a set) assuming chaotic, Cantor set-like structure. Instead of developing local theory, we utilize L^∞ estimates on the complexified solutions via a plurisubharmonic measure type maximum principle. In fact, we prove a more general result, namely that $u \in L^\infty(0, T_s; L^\alpha_w(\Omega))$, coupled with the Cantor set-like geometry implies that $d_H(S_{T_s}) \leq 3 - \alpha$, for all $2 \leq \alpha < 3$. We would like to point out that chaotic structure of the blow-up profile is a physically interesting case in the sense that some theories explain 3D turbulence via the existence of a chaotic singular set [L], [M].

The paper is organized as follows. In Chapter 2, we recall some analyticity properties of 3D NSE, as well as some basic estimates. Chapter 3 contains a plurisubharmonic measure-type maximum principle, and Chapter 4 the main result.

1991 *Subject Classification:* 35Q30, 76D03.
Key words and phrases: Navier-Stokes equations, singular set, turbulence.

Submitted August 20, 1999. Published December 3, 1999.

2. Some known properties of solutions

We consider normalized (unit viscosity, period = 2π) NSE in $\Omega = [0, 2\pi]^3$ with periodic boundary conditions and potential force.

$$\frac{\partial u}{\partial t} - \triangle u + (u \cdot \nabla)u + \nabla \pi = 0\,, \qquad \nabla \cdot u = 0 \tag{2.1}$$

for $(x,t) \in \Omega \times (0,\infty)$, supplemented with the initial condition

$$u(x,0) = u_0(x),\ x \in \Omega,$$

where the $\mathbb{R}^3$-valued function u is the velocity and the $\mathbb{R}$-valued function π is the pressure. Also, we require that

$$\int_\Omega u(x,t)\ dx = 0,\ t \geq 0, \tag{2.2}$$

and $\nabla \cdot u_0 = 0$.

A self-contained presentation of various aspects of the mathematical theory of the NSE can be found in [CF].

The basic energy estimate for the NSE is obtained (formally) multiplying the equations by u in $L^2(\Omega)$. Integrating by parts, utilizing $\nabla \cdot u = 0$ and applying Poincaré inequality one arrives at

$$\|u(t)\|_{L^2}^2 \leq \|u_0\|_{L^2}^2 e^{-ct}\,, \tag{2.3}$$

for all $t > 0$, i.e. L^2 norm decays exponentially in time. For a weak solution, exponential decay is valid only for a large enough t; however,

$$\|u(t)\|_{L^2} \leq \|u_0\|_{L^2} \tag*{(2.3)$'$}$$

is valid *a.e.* in t.

Let now $M, t > 0$ and denote by $\Omega_t(M)$ a super-level set

$$\{x \in \Omega :\ |u(x,t)| \geq M\}.$$

Then, (2.3)$'$ implies the following weak-L^2 estimate on any weak solution of (2.1).

$$\lambda^3(\Omega_t(M)) \leq \frac{\|u_0\|_{L^2}^2}{M^2}\,, \tag{2.4}$$

for all $M > 0$, *a.e.* in t, where λ^3 denotes Lebesgue measure on $\mathbb{R}^3$.

The following theorem is an L^∞ description of local in time analytic smoothing of 3D NSE. H^1- version was previously obtained in [FT2] via Gevrey-class technique.

Theorem 2.1 [GK]. *Let $T = 1/\big(c\|u_0\|_{L^\infty}^2(1+\log_+\|u_0\|_{L^\infty})^2\big)$. Then, the solution u of (2.1), (2.2) on (0,T) satisfies the following property: for every $t \in (0,T)$, u is a restriction of an analytic function $u(x,y,t) + iv(x,y,t)$ in the region*

$$\mathcal{R}_t = \{x + iy \in \mathbb{C}^3 : |y| \leq c^{-1}t^{1/2}\}.$$

Moreover, there exists an absolute constant K such that

$$\|u(\cdot,y,t)\|_{L^\infty} + \|v(\cdot,y,t)\|_{L^\infty} \leq K\|u_0\|_{L^\infty}\,,$$

for $t \in (0,T)$ and $(x,y) \in \mathcal{R}_t$.

3. A plurisubharmonic measure type maximum principle in $\mathbb{C}^n$

The following $\mathbb{C}^n$ version of "2-constants" theorem for product domains follows from a general theorem that is expressed in terms of plurisubharmonic measures [Ga] and the fact that for product domains plurisubharmonic measure can be expressed in terms of one-dimensional harmonic measures [GaKa].

Theorem 3.1. *Let $D_1, D_2, \ldots, D_n$ be open sets in $\mathbb{C}$, and let $E_1, E_2, \ldots, E_n$ be such that $E_i \subset \partial D_i$, for $i = 1, 2, \ldots, n$. Denote by ω_{E_i} harmonic measures of E_i with respect to D_i, and assume that f is a bounded analytic function on $D = D_1 \times D_2 \times \ldots \times D_n$ satisfying the following property:*

$$\|f\|_{L^\infty(E_i)} \leq M_2,$$

for $i = 1, 2, \ldots, n$, and

$$\|f\|_{L^\infty(\complement E_i)} \leq M_1,$$

for $i = 1, 2, \ldots, n$ (the inequalities are in the sense of lim sup *through the interior of D_i). Then*

$$|f(z)| \leq M_2^{\inf(\omega_{E_1}(z), \omega_{E_2}(z), \ldots, \omega_{E_n}(z))} M_1^{1-\inf(\omega_{E_1}(z), \omega_{E_2}(z), \ldots, \omega_{E_n}(z))},$$

for all $z \in D$.

4. Uniform Cantor sets and the main result

We start with a standard construction of a uniform one-dimensional Cantor set (c.f. [F]).

Let $m \geq 2$ be an integer and $s > 1$. Start with an interval $[0, L]$, and then construct a uniform Cantor set $C_{m,s}$ inductively in the following way. In every step, each basic interval I is replaced by m equally spaced subintervals of lengths $\frac{1}{sm} I$, the ends of I coinciding with the ends of the extreme subintervals. The limit set is $C_{m,s}$.

Using standard techniques, one can prove the following result.

Proposition 4.1. *Let $C_{m,s}$ be an one-dimensional uniform Cantor set with parameters $m \in \mathbb{N}$, $m \geq 2$ and $s > 1$. Then*

$$d_H C_{m,s} = \frac{\log m}{\log s + \log m}.$$

Remark 4.2 For every fixed m, the range of Hausdorff dimension of $C_{m,s}$ is $(0, 1)$, i.e. it covers all fractal dimensions.

We will construct our three-dimensional Cantor sets as products of uniform one-dimensional Cantor sets.

The estimate in Theorem 3.1 implies that the worst case scenario is when all component sets have the same harmonic measure, and so we will assume that all three one-dimensional Cantor sets in the product are of the same type. Hence, our singular set in space $S_{m,s}$ will have the form

$$S_{m,s} = C_{m,s} \times C_{m,s} \times C_{m,s}. \tag{4.1}$$

Although Hausdorff dimension of a product is not always equal to the sum of Hausdorff dimensions of the components, it is true if the Hausdorff and the upper box dimensions of the component sets coincide. One can easily check that for one-dimensional uniform Cantor sets, and thus Proposition 4.1 implies the following.

Proposition 4.3. *Let $S_{m,s}$ be the set defined in (4.1). Then*

$$d_H S_{m,s} = \frac{3\log m}{\log s + \log m}.$$

Remark 4.4 Two-parameter family of the sets $S_{m,s}$ is a reasonable model for a chaotic singular set - the Hausdorff dimension has the range $(0,3)$.

A weak-type estimate (e.g. (2.4)) imposes a decay rate on the super-level sets $\Omega_t(M)$. On the other hand, we assumed a chaotic structure of the singular set which imposes a dispersion of the sets $\Omega_t(M)$. The idea is to combine these two properties via Theorem 3.1 - shortly, harmonic measure should eliminate a blow-up scenario in which the singular set is too chaotic, i.e. if $d_H S_{m,s}$ is too big.

Before we precisely formulate our assumption on the geometry of the blow-up profile, we recall the following local in time existence and uniqueness result.

Theorem 4.5 [L]. *Let $\nabla u_0 \in L^2(\Omega)$. Then, there exists $T^*(\|\nabla u_0\|_{L^2}) > 0$ such that (1.1), (1.2) has a unique regular solution on $(0, T^*)$.*

Start with the initial data $u_0, \|\nabla u_0\|_{L^2} < \infty$, and let $T_s \geq T^*$ be the first (possible) singular time. Assume that $S_{T_s} = S_{m,s}$, and consider the following build-up of the flow compatible with the standard construction of $S_{m,s}$.

Let $0 < \epsilon < T_s$, and let $[T_s - \epsilon, T_s) = \cup_{k=1}^{\infty} I_k$, where $\{I_k\}_{k=1}^{\infty}$ is a disjoint decomposition of $[T_s - \epsilon, T_s)$ in the intervals $I_k = [a_k, b_k)$, $(b_k - a_k) \to 0$, $k \to \infty$.

Denote by $S_{m,s}^k$ the kth generation in the standard construction of $S_{m,s}$, and assume that for $\tau \in I_k$,

$$\Omega_\tau(M) \subset S_{m,s}^k, \tag{A1}$$

and

$$\frac{1}{c}(\frac{1}{s})^k \leq \lambda^1(\Pi^i(\Omega_\tau(M))), \tag{A2}$$

$i = 1, 2, 3$, all M satisfying

$$\frac{1}{K^4}\|u(\tau)\|_{L^\infty} \leq M \leq \frac{1}{K^3}\|u(\tau)\|_{L^\infty}.$$

Above, $c \geq 1$ is a suitable absolute constant, $K > 1$ is the absolute constant from Theorem 2.1, λ^1 denotes 1D Lebesgue measure, and Π^i the projection on the ith coordinate.

Remark 4.6 $2\pi(1/s)^k$ is the linear measure of the kth generation in the standard construction of $S_{m,s}$.

We are now ready to state the main result.

Theorem 4.7. *Let $\nabla u_0 \in L^2(\Omega)$, and let T_s be the first (possible) singular time. Assume that the blow-up profile is given by (A1)-(A2), and $u \in L^\infty(0,T_s;L^\alpha_w(\Omega))$, for some $2 \le \alpha < 3$. Then*

$$d_H S_{T_s} \le 3 - \alpha.$$

Remark 4.8 $u \in L^\infty(0,T_s;L^2_w(\Omega))$ is not an assumption - it follows from (2.4).

Proof: We argue by contradiction. Assume that $d_H S_{T_s} = 3 - \alpha + \eta$, for some $0 < \eta < \alpha$. Then, a simple calculation utilizing Proposition 4.3 implies the following relation between the parameters s, m,

$$s = (sm)^{(\alpha-\eta)/3}. \tag{4.2}$$

Translating the estimate from Theorem 2.1 in time, we obtain that for any $t \in (0,T_s)$, the solution u is a restriction of an analytic function $u = u + iv$ with the uniform radius of analyticity at time

$$\tau(t) = t + \frac{c}{\|u(t)\|^2_{L^\infty}(\log \|u(t)\|_{L^\infty})^2} \tag{4.3}$$

at least

$$\rho(\tau(t)) = \frac{c}{\|u(t)\|_{L^\infty}(\log \|u(t)\|_{L^\infty})}. \tag{4.4}$$

Also,

$$\|u(\tau(t))\|_{L^\infty(\{z=x+iy\in\mathbb{C}^3:\ |y|\le\rho(\tau(t))\})} \le K\|u(t)\|_{L^\infty}. \tag{4.5}$$

Let $t \in [T_s - \epsilon, T_s)$ be an "escaping time", i.e. $\|u(t')\|_{L^\infty} > \|u(t)\|_{L^\infty}$, for all $t < t' < T_s$, and define

$$M(t) = \frac{1}{K^3}\|u(t)\|_{L^\infty}.$$

Consider now $\tau = \tau(t)$, where $\tau(t)$ is given by (4.3). Then, $\tau \in I_k$ for some $k \in \mathbb{N}$, and one can easily check (using (4.5) and the fact that t is an escaping time) that $M(t)$ satisfies both inequalities required in our geometric assumption.

Hence, $(A1) - (A2)$ coupled with $u \in L^\infty(0,T_s;L^\alpha_w(\Omega))$ (we can choose an escaping time t such that L^α_w bound holds at τ) imply

$$(\frac{1}{s})^k \le c\frac{1}{M(t)^{\alpha/3}}. \tag{4.6}$$

Inserting (4.2), we arrive at

$$(\frac{1}{c}M(t))^{\frac{\alpha}{\alpha-\eta}} \le (sm)^k. \tag{4.7}$$

Since $\frac{\alpha}{\alpha-\eta} > 1$, for every $c^* > 0$, there exists $M^*(c^*; \alpha, \eta) > 0$ such that

$$c^* M(t) \log M(t) \leq (sm)^k, \tag{4.8}$$

for all $M(t) \geq M^*$. c^* will be chosen later in the proof. Notice that we can always choose an escaping time t such that $M(t)$ is large enough.

To be able to successfully apply Theorem 3.1, harmonic measure

$$\omega_{\Pi^1((\Omega_{\tau(t)}(M(t)))_{per})}(w), \tag{4.9}$$

with respect to

$$D^1_{\tau(t)} = \{z_1 = x_1 + iy_1 \in \mathbb{C} : \ 0 \leq y_1 \leq \rho(\tau(t))\}$$

(we could work with any coordinate projection), computed at

$$w \in \{z_1 = x_1 + iy_1 \in \mathbb{C} : \ y_1 = \frac{\rho(\tau(t))}{2}\}$$

should stay uniformly bounded away from $\frac{1}{2}$.

Since

$$\rho(\tau(t)) = \frac{c}{\|u(t)\|_{L^\infty} (\log \|u(t)\|_{L^\infty})} = \frac{c}{K^3 M(t) \ \log(K^3 M(t))} \geq \frac{c}{M(t) \ \log(M(t))}, \tag{4.10}$$

for $M(t)$ large enough, and since the length of an interval in $S^k_{m,s}$ is

$$2\pi (\frac{1}{sm})^k,$$

an elementary harmonic measure computation (taking into account $(A1)$) implies that it is enough to require that

$$\frac{M(t) \log M(t)}{(sm)^k} \leq \frac{1}{c^*}, \tag{4.11}$$

for a sufficiently large $c^*(m, s)$ (this is true for all $k \geq k^*(s)$, and we can always assume that k is large enough). More precisely, there exists $c^*(m, s)$ such that (4.11) implies

$$\omega_{\Pi^1((\Omega_{\tau(t)}(M(t)))_{per})}(w) \leq \frac{1}{4}, \tag{4.12}$$

for all $w \in \{z_1 = x_1 + iy_1 \in \mathbb{C} : \ y_1 = \frac{\rho(\tau(t))}{2}\}$.

Harmonic measure condition (4.11) is equivalent to (4.8), and hence satisfied for an appropriate escaping time t.

Consider now positive part of the domain of analyticity of $u(\tau(t))$,

$$D_{\tau(t)} = \{z = x + iy \in \mathbb{C}^3 : \ 0 \leq y_i \leq \rho(\tau(t)), \ i = 1, 2, 3\}.$$

Then, the estimate (4.5) implies

$$\|u(\tau(t))\|_{L^\infty(\partial D_{\tau(t)})} \leq K^4 M(t). \tag{4.13}$$

Also,

$$|u(\tau(t), z)| \leq M(t), \tag{4.14}$$

for $z \in G_{\tau(t)} = \partial D_{\tau(t)} - \{\{z = x + iy \in \mathbb{C}^3 : \ y_i = \rho(\tau(t)), \ i = 1, 2, 3\} \cup (\Omega_{\tau(t)}(M(t)))_{per}\}$.

Since the harmonic measure of the projection of $B_{\tau(t)} = \partial D_{\tau(t)} - G_{\tau(t)}$ is by (4.12) less or equal to $\frac{3}{4}$ uniformly in a set containing $P_{\tau(t)} = \{z = x + iy \in \mathbb{C}^3 : y_i = \frac{\rho(\tau(t))}{2}, \ i = 1, 2, 3\}$, Theorem 3.1 yields

$$\|u(\tau(t)\|_{L^\infty(P_{\tau(t)})} \leq \left(K^4 M(t)\right)^{3/4} M(t)^{1/4} = \|u(t)\|_{L^\infty}. \tag{4.15}$$

By symmetry $(u(\overline{z}) = \overline{u(z)})$, the same estimate holds on negative part of domain $D_{\tau(t)}$ as well, and thus the maximum principle gives

$$\|u(\tau(t))\|_{L^\infty} \leq \|u(t)\|_{L^\infty}, \tag{4.16}$$

contradicting t being an escaping time. □

Acknowledgments The author thanks Professors L. Caffarelli and C. Foias for their interest and stimulating discussions.

References

[CF] P. Constantin and C. Foias, "Navier-Stokes equations", Chicago Lectures in Mathematics, Chicago/London, 1988.

[CKN] L. Caffarelli, R. Kohn and L. Nirenberg, *Partial regularity of suitable weak solutions of the Navier-Stokes equations*, Comm. Pure Appl. Math. 35 (1982), 771–831.

[F] K. Falconer, Fractal Geometry, Mathematical Foundations and Applications, *John Wiley and Sons Ltd.*, 1990.

[FT1] C. Foias and R. Temam, *Some analytic and geometric properties of the solutions of the evolution Navier-Stokes equations*, J. Math. Pures et Appl. 58 (1979), 339–368.

[FT2] C. Foias and R. Temam, *Gevrey class regularity for the solutions of the Navier-Stokes equations*, J. Funct. Anal. 87 (1989), 359–369.

[Ga] B. Gaveau, *Estimations des mesures plurisousharmoniques et des spectres associés*, C.R. Acad. Sc., Série A 288 (1979), 969–972.

[GaKa] B. Gaveau and J. Kalina, *Calculs explicites de mesures plurisousharmoniques et des feuilletages associés*, Bull. Sc. Math., 2^e Série 108 (1984), 197–223.

[GK] Z. Grujić and I. Kukavica, *Space analyticity for the Navier-Stokes and related equations with initial data in L^p*, J. Funct. Anal. 152 (1998), 447–466.

[L] J. Leray, *Sue le mouvement d'un liquide visqueux emplissant l'espace*, Acta Mathematica 63 (1934), 193–248.

[Li] F. - H. Lin, *A new proof of the Caffarelli-Kohn-Nirenberg Theorem*, Comm. Pure Appl. Math 51(3) (1998), 241–257.

[M] B. Mandelbrot, The Fractal Geometry of Nature, *Freeman, San Francisco*, 1982.

[Sa] A. Sadullaev, *Plurisubharmonic measures and capacities on complex manifolds,* Russian Math. Surveys 36:4 (1981), 61–119.

[Sch1] V. Scheffer, *Partial regularity of solutions to the Navier-Stokes equations*, Pacific J. Math. 66(2) (1976), 535–552.

[Sch2] V. Scheffer, *Hausdorff measure and the Navier-Stokes equations*, Comm. Math. Phys. 55(2) (1977), 97-112

[St] M. Struwe, *On partial regularity results for the Navier-Stokes equations*, Comm. Pure Appl. Math. 41(4) (1988), 437–458.

Zoran Grujić
Department of Mathematics
University of Texas
Austin, TX 78712, USA
e-mail: grujic@math.utexas.edu