

## LOCAL WELL-POSEDNESS OF THE GENERALIZED KORTEWEG-DE VRIES EQUATION IN SPACES OF ANALYTIC FUNCTIONS

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**Abstract.** Local well-posedness for the generalized KdV equation is obtained in a class of functions analytic on a strip around the real axis *without shrinking* the width of the strip in time. The proof relies on space-time estimates that previously have been used mainly for low-regularity spaces.

### 1. INTRODUCTION

Consideration is given to the local well-posedness of the Cauchy problem for the generalized Korteweg-de Vries equation (GKdV)

$$u_t + u^p u_x + u_{xxx} = 0, \quad u(x, 0) = u_0(x), \quad (1.1)$$

with initial data  $u_0$  in a class of functions analytic in a symmetric strip around the real axis. The number  $p$  is taken to be a positive integer. A number of authors have given results concerning this problem. Kato and Masuda [10] developed a method of obtaining spatial analyticity of solutions for a large class of evolution equations, and applied it to the GKdV equation assuming analytic initial data and boundedness of a solution in  $H^\infty = \bigcap_{s>0} H^s$ , where  $H^s$  denotes the standard  $L^2$ -based Sobolev space of order  $s$ . Their result implies that the solution will stay analytic in a strip, although the width of the strip may *decrease* as time advances. More recently, Hayashi [8] proved a local-in-time analyticity result without assuming any *a priori* regularity, but the width of the strip was also allowed to *decrease* in time. The same type of result can be obtained utilizing analytic Gevrey

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norms on the spaces  $G^{\sigma,s}$  which will be defined in a moment. For another application of Gevrey-class analysis to a large class of KdV-type equations see [1] where it is observed that the loss of analyticity detects the loss of  $L^\infty$ -regularity, i.e. a finite-time blow-up. This analysis is relevant for  $p \geq 4$ , a regime in which the possibility of a finite-time blow-up is expected. In the aforementioned approaches the decrease of the width of the strip is essential in order to compensate for the derivative loss in the nonlinearity. In [9], Hayashi proved local well-posedness in a class of analytic functions that was more in the style of the well-posedness results for the GKdV in the usual Sobolev spaces. Hayashi worked in a class of functions analytic on a double cone symmetric around the real axis, for which there are nice isometric identities among various derivatives. Using Kato's smoothing effect inductively on all derivatives, he was able to prove a local well-posedness result *without shrinking* the cone angle in time. Since the identities he used are very special to cone-type domains, it appears that his approach could not be applied to a strip.

The goal of this note is to show that it is possible to obtain local well-posedness for a strip *without shrinking* the width of the strip in time. The proof relies on a version of Kato's smoothing effect, and subsequently a multilinear estimate in a class of functions analytic on a strip. In order to overcome the derivative loss, use is made of special function spaces which are similar to the spaces introduced by Bourgain [2], and later extensively used by several authors (see, e.g., [3, 4, 5, 7, 11, 12]). So far, the Bourgain-type spaces have been used primarily to obtain well-posedness in low-regularity classes, such as Sobolev spaces with negative indices. Here, they find an application on the opposite end of the scale in analytic classes.

Before leaving the introduction, it is convenient to state the main result of this article. For  $\sigma > 0$  and  $s \in \mathbb{R}$ , define  $G^{\sigma,s}$  to be the subspace of  $L^2(\mathbb{R})$  for which

$$\|u_0\|_{G^{\sigma,s}}^2 = \int (1 + |\xi|)^{2s} e^{2\sigma(1+|\xi|)} |\hat{u}_0(\xi)|^2 d\xi$$

is finite.

**Theorem 1.** *Let  $s \geq 0$  when  $p = 1$  and  $s > \frac{3}{2}$  when  $p \geq 2$ . For initial data in  $G^{\sigma,s}$ ,  $\sigma > 0$ , there exists a positive time  $T$ , such that the initial-value problem (1.1) is well-posed in the space  $C([0, T], G^{\sigma,s})$ .*

Well-posedness is understood in the usual Hadamard-sense, including existence, uniqueness and continuous dependence on the initial data.

$C([0, T], G^{\sigma,s})$  denotes the space of continuous functions from the time interval  $[0, T]$  into  $G^{\sigma,s}$ . Membership in this class is equivalent to the requirement that the solution describe a continuous curve in the solution space, a property which is sometimes called the persistence property. The value of  $s$  is of minor significance as the classes considered in this theorem are spaces of analytic functions. The paper is organized as follows. In Section 2, we define the functional spaces and recall some known results. Section 3 contains the proof of the crucial multilinear estimates, and in Section 4, Theorem 1 is proved via a contraction argument.

## 2. FUNCTIONAL SETTING AND LINEAR ESTIMATES

The Fourier transform of a function  $u_0$  belonging to the Schwartz class  $\mathcal{S}$  is defined by

$$\hat{u}_0(\xi) = \frac{1}{\sqrt{2\pi}} \int u_0(x)e^{-ix\xi} dx.$$

The unadorned integral denotes integration over the real line  $\mathbb{R}$ . A class of analytic functions suitable for the analysis in this paper is the analytic Gevrey class [6]. Defining a Fourier multiplier operator  $A$  by

$$\widehat{Au}_0(\xi) = (1 + |\xi|)\hat{u}_0(\xi),$$

the Gevrey norm of order  $(\sigma, s)$  can be written as

$$\|u_0\|_{G^{\sigma,s}} = \|A^s e^{\sigma A} u_0\|_{L^2(\mathbb{R})}.$$

It is straightforward to check that a function in  $G^{\sigma,s}$  is a restriction to the real axis of a function analytic on a symmetric strip of width  $2\sigma$ . As mentioned in the introduction, in order to prove the local well-posedness result, we have to introduce another family of function spaces. For  $\sigma > 0$ ,  $s \in \mathbb{R}$ , and  $b \in [-1, 1]$ , define  $X_{\sigma,s,b}$  to be the Banach space equipped with the space-time norm

$$\|v\|_{\sigma,s,b}^2 = \iint (1 + |\tau - \xi^3|)^{2b} (1 + |\xi|)^{2s} e^{2\sigma(1+|\xi|)} |\hat{v}(\xi, \tau)|^2 d\xi d\tau.$$

In what follows, for a function  $v(x, t)$  of two variables, the notation  $\hat{v}(\xi, \tau)$  is used to denote the space-time Fourier transform. The space-time symbol in the  $X_{\sigma,s,b}$ -norm is adapted to the linear part of the equation. In the following, some identities and linear estimates are listed that reveal this relation in an explicit fashion. For the initial-value problem

$$w_t + w_{xxx} = 0, \quad w(x, 0) = w_0(x), \tag{2.1}$$

the Fourier transform can be used to give an explicit solution in terms of the propagator  $W(t)$ ,

$$w(x, t) = W(t)w_0 = c \int e^{ix\xi} e^{it\xi^3} \widehat{w}_0(\xi) \, d\xi. \tag{2.2}$$

Defining the operators  $\Lambda^\rho$ ,  $\rho \in \mathbb{R}$ , by

$$\widehat{\Lambda^\rho v}(\xi, \tau) = (1 + |\tau|)^\rho \widehat{v}(\xi, \tau),$$

we have the following identity:

$$\|Wv\|_{\sigma,s,b} = \|A^s e^{\sigma A} \Lambda^b v\|_{L^2(\mathbb{R}^2)}.$$

As already mentioned, the space of continuous functions on the interval  $[0, T]$  with values in  $G^{\sigma,s}$  is denoted by  $C([0, T], G^{\sigma,s})$ . This space is a Banach space when equipped with the norm

$$\|u\|_{C_{T,\sigma,s}} = \sup_{0 \leq t \leq T} \|u(\cdot, t)\|_{G^{\sigma,s}}.$$

If  $b > \frac{1}{2}$ ,  $X_{\sigma,s,b}$  is embedded in  $C([0, T], G^{\sigma,s})$ . In fact, the inequality

$$\|u\|_{C_{T,\sigma,s}} \leq c \|u\|_{\sigma,s,b} \tag{2.3}$$

follows directly from the Sobolev Embedding Theorem. Note that for  $\sigma = 0$ , the space  $X_{\sigma,s,b}$  coincides with the space  $X_{s,b}$  introduced by Bourgain. We close this section by recalling some linear estimates which hold for these spaces. Let  $\psi$  be an infinitely differentiable function on  $\mathbb{R}$ , such that

$$\psi(t) = \begin{cases} 0, & |t| \geq 2, \\ 1, & |t| \leq 1, \end{cases}$$

and let  $\psi_T(t) = \psi(t/T)$ .

**Lemma 1.** *Let  $\sigma > 0$ ,  $b > \frac{1}{2}$ , and  $b - 1 < b' < 0$ . Then there is a constant  $c$  such that the following estimates hold.*

$$\|\psi(t) W(t)u_0(x)\|_{\sigma,s,b} \leq c \|u_0\|_{G^{\sigma,s}}, \tag{2.4}$$

$$\left\| \psi_T(t) \int_0^t W(t-s)v(s) \, ds \right\|_{\sigma,s,b} \leq c T^{1-b+b'} \|v\|_{\sigma,s,b'}. \tag{2.5}$$

With the same assumptions as in the lemma, but with  $\sigma = 0$ , (2.4) was proved in [11], and (2.5) was proved in [7]. These inequalities clearly remain valid for  $\sigma > 0$ , as one merely has to replace  $u_0$  by  $e^{\sigma A}u_0$  and  $v$  by  $e^{\sigma A}v$  in these results.

3. A MULTILINEAR ESTIMATE

In this section estimates for the nonlinear term in the equation are proved. These results are similar to the estimates of Bourgain and Kenig, Ponce, and Vega. In fact, in the quadratic case, the estimate reduces to the bilinear estimate in [12].

**Theorem 2.** *Suppose  $u$  and  $v$  are in  $X_{\sigma,s,b}$ , where  $s \geq 0$ ,  $\sigma > 0$ , and  $\frac{1}{2} < b < \frac{3}{4}$ . If  $b - 1 < b' < -\frac{1}{4}$ , then there exists a constant  $c$  depending only on  $s, b$ , and  $b'$  such that*

$$\|\partial_x(uv)\|_{\sigma,s,b'} \leq c \|u\|_{\sigma,s,b} \|v\|_{\sigma,s,b}. \tag{3.1}$$

**Proof.** Only the case  $s = 0$  is treated. When  $s > 0$  the proof follows immediately from the case  $s = 0$ . Note that (3.1) can be written more explicitly as

$$\left\| (1 + |\tau - \xi^3|)^{b'} e^{\sigma(1+|\xi|)} \widehat{\partial_x(uv)}(\xi, \tau) \right\|_{L_\xi^2 L_\tau^2} \leq c \|u\|_{\sigma,0,b} \|v\|_{\sigma,0,b}, \tag{3.2}$$

or

$$\left\| (1 + |\tau - \xi^3|)^{b'} e^{\sigma(1+|\xi|)} \xi (\hat{u} * \hat{v})(\xi, \tau) \right\|_{L_\xi^2 L_\tau^2} \leq c \|u\|_{\sigma,0,b} \|v\|_{\sigma,0,b}. \tag{3.3}$$

Now note that if we let

$$f(\xi, \tau) = ((1 + |\tau - \xi^3|)^b e^{\sigma(1+|\xi|)} \hat{u})(\xi, \tau)$$

and

$$g(\xi, \tau) = ((1 + |\tau - \xi^3|)^b e^{\sigma(1+|\xi|)} \hat{v})(\xi, \tau),$$

then (3.3) is equivalent to

$$\left\| \frac{\xi e^{\sigma(1+|\xi|)}}{(1 + |\tau - \xi^3|)^{-b'}} \iint \frac{e^{-\sigma(1+|\xi_1|)} f(\xi_1, \tau_1)}{(1 + |\tau_1 - \xi_1^3|)^b} \frac{e^{-\sigma(1+|\xi-\xi_1|)} g(\xi - \xi_1, \tau - \tau_1)}{(1 + |\tau - \tau_1 - (\xi - \xi_1)^3|)^b} d\xi_1 d\tau_1 \right\|_{L_\xi^2 L_\tau^2} \leq c \|f\|_{L_\xi^2 L_\tau^2} \|g\|_{L_\xi^2 L_\tau^2}.$$

A proof of this estimate can be obtained by duality. Let  $d(\xi, \tau)$  be a positive function in  $L^2(\mathbb{R}^2)$  with norm  $\|d\|_{L^2(\mathbb{R}^2)} = 1$ . Then we need to estimate

$$\int_{\mathbb{R}^4} \frac{d(\xi, \tau) |\xi| e^{\sigma(1+|\xi|)}}{(1 + |\tau - \xi^3|)^{-b'}} \frac{f(\xi_1, \tau_1) e^{-\sigma(1+|\xi_1|)}}{(1 + |\tau_1 - \xi_1^3|)^b} \frac{g(\xi - \xi_1, \tau - \tau_1) e^{-\sigma(1+|\xi-\xi_1|)}}{(1 + |\tau - \tau_1 - (\xi - \xi_1)^3|)^b} d\mu, \tag{3.4}$$

where  $d\mu = d\xi_1 d\tau_1 d\xi d\tau$ . Now observe that using the triangle inequality  $|\xi| \leq |\xi_1| + |\xi - \xi_1|$ , (3.4) can be dominated by

$$\int_{\mathbb{R}^4} \frac{d(\xi, \tau) |\xi|}{(1 + |\tau - \xi^3|)^{-b'}} \frac{f(\xi_1, \tau_1)}{(1 + |\tau_1 - \xi_1^3|)^b} \frac{g(\xi - \xi_1, \tau - \tau_1)}{(1 + |\tau - \tau_1 - (\xi - \xi_1)^3|)^b} d\mu.$$

This shows that the proof in fact reduces to the proof of Kenig, Ponce, and Vega given in [12].  $\square$

The bilinear estimate coupled with the linear estimates in the previous section suffices to prove the local well-posedness of the KdV ( $p = 1$ ) in the analytic classes  $X_{\sigma,s,b}$ . To be able to treat the GKdV ( $p \geq 2$ ) in a similar fashion, a multilinear analog of the bilinear estimate is needed. As pointed out in [12], (already) a trilinear estimate fails in  $X_{s,b}$  for any  $s < \frac{1}{4}$ . However, since we work in analytic classes  $X_{\sigma,s,b}$ , the value of  $s$  plays no role, and one may allow positive values of  $s$ .

**Theorem 3.** *Let  $b > \frac{1}{2}$ ,  $b' < -\frac{1}{4}$ , and  $s \geq 3b$ . Let  $p \in \mathbb{N}$ , and suppose  $u_1, u_2, \dots, u_{p+1} \in X_{\sigma,s,b}$ . Then there exists a constant  $c$  depending only on  $p, s, b$ , and  $b'$  such that*

$$\|\partial_x(\prod_{i=1}^{p+1} u_i)\|_{\sigma,s,b'} \leq c \prod_{i=1}^{p+1} \|u_i\|_{\sigma,s,b}. \tag{3.5}$$

The proof of Theorem 3 will be established using a number of auxiliary results. To state these results, we first need to introduce some more notation. For  $\rho \in \mathbb{R}$ , and a suitable  $f$ , define  $F_\rho$  by its Fourier transform  $\widehat{F}_\rho$ ,

$$\widehat{F}_\rho(\xi, \tau) = \frac{f(\xi, \tau)}{(1 + |\tau - \xi^3|)^\rho}.$$

The first lemma is a version of Kato’s smoothing effect proved by Bourgain [2].

**Lemma 2.** *For  $\rho > \frac{1}{4}$ , there exists a constant  $c$  depending only on  $\rho$  such that*

$$\|A^{\frac{1}{2}} F_\rho\|_{L_x^4 L_t^2} \leq c \|f\|_{L_\xi^2 L_\tau^2}. \tag{3.6}$$

The next two lemmas provide control over the  $L_x^\infty L_t^\infty$  and  $L_x^2 L_t^\infty$ -norms.

**Lemma 3.** *For  $\rho > \frac{1}{2}$ , and  $s > \frac{1}{2}$ , there exists a constant  $c$  depending only on  $\rho$  and  $s$ , such that*

$$\|A^{-s} F_\rho\|_{L_x^\infty L_t^\infty} \leq c \|f\|_{L_\xi^2 L_\tau^2}. \tag{3.7}$$

**Proof.** The proof follows from the Riemann-Lebesgue Lemma and Hölder’s inequality.

$$\begin{aligned} \|A^{-s} F_\rho\|_{L_x^\infty L_t^\infty} &\leq c \int \int \frac{|f(\xi, \tau)|}{(1 + |\tau - \xi^3|)^\rho (1 + |\xi|)^s} d\tau d\xi \\ &\leq c \int \left\{ \int \frac{|f(\xi, \tau)|^2}{(1 + |\xi|)^{2s}} d\tau \right\}^{\frac{1}{2}} \left\{ \int \frac{d\tau}{(1 + |\tau - \xi^3|)^{2\rho}} \right\}^{\frac{1}{2}} d\xi \end{aligned}$$

$$\leq c \left\{ \int \int |f(\xi, \tau)|^2 d\tau d\xi \right\}^{\frac{1}{2}} \left\{ \int \frac{d\xi}{(1 + |\xi|)^{2s}} \right\}^{\frac{1}{2}} \leq c \|f\|_{L_\xi^2 L_\tau^2}.$$

**Lemma 4.** For  $\rho > \frac{1}{2}$ , and  $s \geq 3\rho$ , there exists a constant  $c$  depending only on  $\rho$  and  $s$ , such that

$$\|A^{-s} F_\rho\|_{L_x^2 L_t^\infty} \leq c \|f\|_{L_\xi^2 L_\tau^2}. \tag{3.8}$$

**Proof.** Using the Sobolev Embedding Theorem, it can be seen that

$$\begin{aligned} \|A^{-s} F_\rho\|_{L_x^2 L_t^\infty} &= \left\| \|A^{-s} F_\rho\|_{L_t^\infty} \right\|_{L_x^2} \leq c \left\| \|\Lambda^\rho A^{-s} F_\rho\|_{L_t^2} \right\|_{L_x^2} \\ &= c \left\| \|\Lambda^\rho A^{-s} F_\rho\|_{L_{x,t}^2} \right\|_{L_x^2} = c \left\| \frac{(1 + |\tau|)^\rho}{(1 + |\xi|)^s (1 + |\tau - \xi^3|)^\rho} f(\xi, \tau) \right\|_{L_{\xi,\tau}^2} \\ &\leq c \left\| \frac{(1 + |\tau|)^\rho}{(1 + |\xi|)^s (1 + |\tau - \xi^3|)^\rho} \right\|_{L_{\xi,\tau}^\infty} \|f(\xi, \tau)\|_{L_{\xi,\tau}^2} \leq c \|f(\xi, \tau)\|_{L_{\xi,\tau}^2}. \end{aligned}$$

The fact that

$$\left\| \frac{(1 + |\tau|)^\rho}{(1 + |\xi|)^s (1 + |\tau - \xi^3|)^\rho} \right\|_{L_{\xi,\tau}^\infty}$$

is bounded can be verified easily. □

We are now ready to prove the multilinear estimate in Theorem 3.

**Proof of Theorem 3.** The case  $p = 1$  was treated in Theorem 2. For simplicity of exposition, the proof is first given for the case  $p = 2$ , after which the proof for a general  $p$  will be more transparent. As in the proof of Theorem 2, we have to estimate an integral of the form

$$\begin{aligned} \int_{\mathbb{R}^6} &\frac{d(\xi, \tau) |\xi|^{1+s} e^{\sigma(1+|\xi|)}}{(1 + |\tau - \xi^3|)^{-b'}} \frac{f(\xi_1, \tau_1) e^{-\sigma(1+|\xi_1|)} (1 + |\xi_1|)^{-s}}{(1 + |\tau_1 - \xi_1^3|)^b} \\ &\times \frac{g(\xi - \xi_2, \tau - \tau_2) e^{-\sigma(1+|\xi-\xi_2|)} (1 + |\xi - \xi_2|)^{-s}}{(1 + |\tau - \tau_2 - (\xi - \xi_2)^3|)^b} \\ &\times \frac{h(\xi_2 - \xi_1, \tau_2 - \tau_1) e^{-\sigma(1+|\xi_2-\xi_1|)} (1 + |\xi_2 - \xi_1|)^{-s}}{(1 + |\tau_2 - \tau_1 - (\xi_2 - \xi_1)^3|)^b} d\mu. \end{aligned}$$

Utilizing the inequality  $|\xi| \leq |\xi_1| + |\xi - \xi_2| + |\xi_2 - \xi_1|$  on the exponentials, we are left with

$$\int_{\mathbb{R}^6} \frac{d(\xi, \tau) |\xi|^{1+s}}{(1 + |\tau - \xi^3|)^{-b'}} \frac{f(\xi_1, \tau_1) (1 + |\xi_1|)^{-s}}{(1 + |\tau_1 - \xi_1^3|)^b} \frac{g(\xi - \xi_2, \tau - \tau_2) (1 + |\xi - \xi_2|)^{-s}}{(1 + |\tau - \tau_2 - (\xi - \xi_2)^3|)^b}$$

$$\times \frac{h(\xi_2 - \xi_1, \tau_2 - \tau_1)(1 + |\xi_2 - \xi_1|)^{-s}}{(1 + |\tau_2 - \tau_1 - (\xi_2 - \xi_1)^3|)^b} d\mu.$$

Now, split the Fourier space into six regions, according to all possible combinations of inequalities such as  $|\xi - \xi_2| \leq |\xi_2 - \xi_1| \leq |\xi_1|$ . In this particular case, the integral can be dominated by the inner product

$$\langle A^{\frac{1}{2}}D_{-b'}, A^{\frac{1}{2}}F_b A^{-s}G_b A^{-s}H_b \rangle,$$

and the estimate continues as follows.

$$\begin{aligned} &\langle A^{\frac{1}{2}}D_{-b'}, A^{\frac{1}{2}}F_b A^{-s}G_b A^{-s}H_b \rangle \\ &\leq c \|A^{\frac{1}{2}}D_{-b'}\|_{L_x^4 L_t^2} \|A^{\frac{1}{2}}F_b\|_{L_x^4 L_t^2} \|A^{-s}G_b\|_{L_x^2 L_t^\infty} \|A^{-s}H_b\|_{L_x^\infty L_t^\infty} \\ &\leq c \|d\|_{L_\xi^2 L_\tau^2} \|f\|_{L_\xi^2 L_\tau^2} \|g\|_{L_\xi^2 L_\tau^2} \|h\|_{L_\xi^2 L_\tau^2} = c \|f\|_{L_\xi^2 L_\tau^2} \|g\|_{L_\xi^2 L_\tau^2} \|h\|_{L_\xi^2 L_\tau^2}, \end{aligned}$$

where Lemma 2,3 and 4 were used in the second step. The other cases follow simply by (at most) interchanging the roles of  $F_b, G_b$ , and  $H_b$ .

The proof in the case of higher nonlinearities  $p \geq 3$  is virtually identical. The only difference is that we need to split the Fourier space in  $(p + 1)!$  regions. For example, a splitting in which all the combinations are dominated by  $|\xi_1|$  will lead to the following estimate.

$$\begin{aligned} &\langle A^{\frac{1}{2}}D_{-b'}, A^{\frac{1}{2}}(F_1)_b A^{-s}(F_2)_b \prod_{i=3}^{p+1}(A^{-s}(F_i)_b) \rangle \\ &\leq c \|A^{\frac{1}{2}}D_{-b'}\|_{L_x^4 L_t^2} \|A^{\frac{1}{2}}(F_1)_b\|_{L_x^4 L_t^2} \|A^{-s}(F_2)_b\|_{L_x^2 L_t^\infty} \prod_{i=3}^{p+1} \|A^{-s}(F_i)_b\|_{L_x^\infty L_t^\infty} \\ &\leq c \|d\|_{L_\xi^2 L_\tau^2} \|u_1\|_{L_\xi^2 L_\tau^2} \|u_2\|_{L_\xi^2 L_\tau^2} \prod_{i=3}^{p+1} \|u_i\|_{L_\xi^2 L_\tau^2} = c \prod_{i=1}^{p+1} \|u_i\|_{L_\xi^2 L_\tau^2}. \end{aligned}$$

#### 4. PROOF OF THEOREM 1

With the estimates provided in the previous section, existence and uniqueness of a solution to the initial-value problem in  $X_{\sigma,s,b}$  can be proved easily using a contraction argument. Consider the integral operator

$$\Gamma(v) = \psi(t)W(t)u_0 - \frac{1}{p+1}\psi_T(t) \int_0^t W(t-s)\partial_x(v^{p+1}(s)) ds. \tag{4.1}$$

Let  $r = \|u_0\|_{G^{\sigma,s}}$ . It will be proved that  $T$  can be chosen so that  $\Gamma$  is a contraction in the ball  $B(2cr) \subset X_{\sigma,s,b}$  of radius  $2cr$  centered at 0.

**Lemma 5.** *There exists a positive time  $T$ , such that the operator  $\Gamma$  as defined in (4.1) is a contraction in the ball  $B(2cr)$ .*

**Proof.** First it is proved that  $\Gamma$  is a mapping on  $B(2cr)$ . Using (2.4) and (2.5), and the nonlinear estimate, we see that

$$\begin{aligned} \|\Gamma(v)\|_{\sigma,s,b} &\leq \|\psi(t)W(t)u_0\|_{\sigma,s,b} + c \left\| \psi_T(t) \int_0^t W(t-s)\partial_x(v^{p+1}(s)) ds \right\|_{\sigma,s,b} \\ &\leq c \|u_0\|_{G^{\sigma,s}} + c T^{1-b+b'} \|\partial_x(v^{p+1}(s))\|_{\sigma,s,b'} \\ &\leq c \|u_0\|_{G^{\sigma,s}} + c T^{1-b+b'} \|v\|_{\sigma,s,b}^{p+1} \leq cr + c T^{1-b+b'} (2cr)^{p+1}. \end{aligned}$$

Choosing

$$T < \left( \frac{1}{2^{p+1}c^{p+1}r^p} \right)^{\frac{1}{1-b+b'}},$$

it is seen that  $\Gamma$  maps  $B(2cr)$  to  $B(2cr)$ . With the same choice of  $T$ , the contraction property can be proved.

$$\begin{aligned} \|\Gamma(v) - \Gamma(\bar{v})\|_{\sigma,s,b} &= c \left\| \psi_T(t) \int_0^t W(t-s)\partial_x(v^{p+1}(s) - \bar{v}^{p+1}(s)) ds \right\|_{\sigma,s,b} \\ &\leq c T^{1-b+b'} \|\partial_x(v^{p+1}(s) - \bar{v}^{p+1}(s))\|_{\sigma,s,b'} \\ &\leq c T^{1-b+b'} (2cr)^p \|v - \bar{v}\|_{\sigma,s,b} \leq \frac{1}{2} \|v - \bar{v}\|_{\sigma,s,b}. \quad \square \end{aligned}$$

Since  $\Gamma$  is a contraction, it follows that  $\Gamma$  has a unique fixed point  $u$  in  $B(2cr)$ . The function  $u$  solves the initial-value problem (1.1).

Next, we show the persistence property. This will follow if it is known that the solution  $u$  belongs to the space  $C([0, T], G^{\sigma,s})$ . But, as mentioned in Section 2,  $X_{\sigma,s,b}$  is continuously embedded in  $C([0, T], G^{\sigma,s})$  as long as  $b > \frac{1}{2}$ . Uniqueness of the solution in  $C([0, T], G^{\sigma,s})$  can be proved by the following standard argument.

**Lemma 6.** *Suppose  $u$  and  $v$  are solutions to (1.1) in  $C([0, T], G^{\sigma,s})$  with  $u(\cdot, 0) = v(\cdot, 0)$  in  $G^{\sigma,s}$ . Then  $u = v$ .*

**Proof.** Let  $e = u - v$ . Then  $e$  satisfies the initial-value problem

$$e_t + u^p u_x - v^p v_x + e_{xxx} = 0, \quad e(x, 0) = 0. \tag{4.2}$$

Multiplying by  $e$  and integrating over  $\mathbb{R}$ , the inequality

$$\frac{d}{dt} \|e\|_{L^2} \leq c \|P(u, u_x, v, v_x)\|_{L^\infty} \|e\|_{L^2} \tag{4.3}$$

appears. Since  $u$  and  $v$  belong to  $G^{\sigma,s}$ , the polynomial term is bounded. Then Gronwall's inequality can be used to see that  $e = 0$ .  $\square$

To prove continuous dependence on the initial data, suppose  $u$  and  $\bar{u}$  are solutions corresponding to initial data  $u_0$  and  $\bar{u}_0$ . Similarly as in the proof of Lemma 5 we arrive at

$$\|u - \bar{u}\|_{\sigma,s,b} \leq c \|u_0 - \bar{u}_0\|_{G^{\sigma,s}} + \frac{1}{2} \|u - \bar{u}\|_{\sigma,s,b}.$$

From this inequality, continuous dependence in  $C([0, T], G^{\sigma,s})$  of the solution on the initial data in  $G^{\sigma,s}$  is immediate.

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