DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **14**, Number **4**, April **2006**

Website: http://aimSciences.org

pp. **837–843**

REGULARITY OF FORWARD-IN-TIME SELF-SIMILAR SOLUTIONS TO THE 3D NAVIER-STOKES EQUATIONS

Zoran Grujić

Department of Mathematics
University of Virginia
Charlottesville, VA 22904

(Communicated by Jerry Bona)

Abstract. Any forward-in-time self-similar (localized-in-space) suitable weak solution to the 3D Navier-Stokes equations is shown to be infinitely smooth in both space and time variables. As an application, a proof of infinite space and time regularity of a class of *a priori* singular small self-similar solutions in the critical weak Lebesgue space $L^{3,\infty}$ is given.

1. **Introduction.** Self-similar solutions to the 3D Navier-Stokes equations (NSE) are given by

$$u(x,t) = \lambda(t)U\big(\lambda(t)x\big) \quad \text{and} \quad p(x,t) = \lambda^2(t)P\big(\lambda(t)x\big)$$

where a divergence-free vector field U and a scalar field P on $\mathbb{R}^3$ satisfy either

$$-\triangle U + aU + a(x\cdot\nabla)U + (U\cdot\nabla)U + \nabla P = 0 \tag{1}$$

corresponding to

$$\lambda(t) = \frac{1}{\sqrt{2a(T-t)}} \text{ for } t < T,$$

or

$$-\triangle U - aU - a(x\cdot\nabla)U + (U\cdot\nabla)U + \nabla P = 0$$

corresponding to

$$\lambda(t) = \frac{1}{\sqrt{2a(T+t)}} \text{ for } t > -T, \tag{2}$$

for some $a > 0$ and $T \geq 0$. The former choice generates backward-in-time self-similar solutions, while the latter generates forward-in-time self-similar solutions to the NSE.

The study of backward self-similar solutions was initiated by Leray [L] who observed that if such a nontrivial solution exists, then

$$\lim_{t\to T^-} \|\nabla u(\cdot,t)\|_2 = \infty,$$

i.e., their existence would necessarily lead to the phenomenon of finite-time blow up. The first rigorous proof of non-existence of Leray's backward self-similar solutions was given in [NRS] utilizing Caffarelli-Kohn-Nirenberg theory (CKN) and a certain form of maximum principle connected to the Leray's equation (1) to show that $U \in L^3 \cap W^{1,2}_{loc}$ implies $U \equiv 0$. It was then shown in [MNPS], with no help from

2000 *Mathematics Subject Classification.* Primary: 35, 76.
Key words and phrases. Navier-Stokes equations, regularity, self-similar solutions.

CKN theory, that $U \in W^{1,2}$ prevents the existence of nontrivial backward self-similar solutions, and in about the same time in [Ts] that $U \in W^{1,2}_{loc}$ itself suffices. In conclusion, nontrivial backward self-similar solutions satisfying the localized energy inequality do not exist.

Forward-in-time self-similar solutions of any physical model are of special interest–heuristically, and some times rigorously, they capture the large-time asymptotics of the model. We focus here on the choice $a = \frac{1}{2}, T = 0$ in (2)–the case when $\lambda(t) = \frac{1}{\sqrt{t}}$. Self-similarity, i.e., the scaling invariance is then expressed as follows. If u and p are solutions to the NSE on $\mathbb{R}^3 \times (0, \infty)$ corresponding to initial data u_0, then $u^\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$ and $p^\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t)$ are solutions corresponding to initial data $u_0^\lambda(x) = \lambda u_0(\lambda x)$, for any $\lambda > 0$. If, in addition, the original initial data u_0 are homogeneous of degree -1, i.e. if $u_0(\lambda x) = \lambda^{-1} u_0(x)$ for all x in $\mathbb{R}^3$, and all $\lambda > 0$, and the problem is considered in a uniqueness class for the Navier-Stokes equations, it follows readily that $u(\lambda x, \lambda^2 t) = \lambda^{-1} u(x,t)$ and $p(\lambda x, \lambda^2 t) = \lambda^{-2} p(x,t)$ for all x in $\mathbb{R}^3$, all $t > 0$, and all $\lambda > 0$. It is worth noting that initial data generating (nontrivial) self-similar solutions are necessarily singular – the homogeneity of degree -1 implies the existence of a singularity at the origin of the order of $\frac{1}{|x|}$. This prevents generating self-similar solutions in the 'usual' spaces, e.g., the Lebesgue spaces L^p for $p \geq 3$, or the Sobolev spaces H^s for $s \geq 1/2$. In fact, a $\frac{1}{|x|}$-type singularity is exactly a critical singularity for the critical spaces L^3 and $H^{1/2}$. The first rigorous construction of self-similar solutions is due to Giga and Miyakawa [GM]. They worked in the Morrey-type spaces of measures and obtained self-similar solutions for the vorticity equation starting from small initial data. More recently, Cannone [C] developed a systematic approach to constructing self-similar solutions in a variety of spaces originating at small initial data–see also [CK, CMP, CP, M, P1, P2]. Some of the constructions (e.g., the ones in suitable homogeneous Besov spaces via Kato's two-norms approach) automatically generate regular solutions, while some others (e.g., Meyer's direct construction in the critical weak Lebesgue space $L^{3,\infty}$) generate *a priori* singular solutions. Since the regularizing effect displayed in the aforementioned results seemed to be tied to smallness of initial data, and also partially motivated by the existence of explicit stationary homogeneous of degree -1 (and hence singular) solutions to the NSE on $\mathbb{R}^3 - \{0\}$, it was conjectured in [CK] that forward self-similar solutions originating from large homogeneous of degree -1 initial data may comprise a suitable universe to look for singular solutions.

In this note we observe that *any* forward self-similar (localized-in-space) suitable weak solution to the NSE is infinitely smooth in both space and time variables–hence showing that, at least in the realm of suitable weak solutions, self-similarity itself prevents the formation of singularities. The proof is a direct consequence of the CKN estimate on the size of a possible singular set in the space-time and self-similarity. Let us note in passing here that any self-similar solution satisfying (global-in-space) Leray's energy inequality is necessarily trivial (this was first observed in [GM])–however, the suitable weak solutions satisfying only the localized energy inequality, as, e.g., the ones constructed in [L-R1] (see also the book [L-R2] for a somewhat different construction), are not incompatible with self-similarity.

As an application, we present a proof of regularity of Meyer's [M] $L^{3,\infty}$ self-similar solutions.

Remark 1. Recall that Meyer's construction is based on continuity of the NSE bilinear form in $C_w([0,\infty), L^{3,\infty})$, resulting in small, mild, solutions living in $C_w([0,\infty), L^{3,\infty})$ with no additional regularity. Hence, these solutions are *a priori* singular.

On the other hand, if one follows Kato's two-norms approach, one does get additional regularity for certain classes of mild solutions originating at sufficiently small $L^{3,\infty}$-initial data [B1, B2, C, KY, Y]. Here, one has to work in a *smaller* space, e.g., in $Z^p = C_w([0,\infty), L^{3,\infty}) \cap X^p$, for some $p > 3$, where

$$X^p = \{u \in C((0,\infty), L^{p,\infty}) \,|\, t^{(1-\frac{3}{p})/2}\|u(t)\|_{p,\infty} \le M < \infty\}.$$

However, this implies *additional* smallness on the initial data, and consequently, these constructions do not encompass all Meyer's solutions.

2. **Preliminaries.** We consider the 3D Navier-Stokes equations with unit viscosity and zero external force,

$$\begin{aligned} &u_t - \triangle u + (u \cdot \nabla)u + \nabla p = 0 \\ &\nabla \cdot u = 0, \end{aligned} \tag{3}$$

where u denotes the velocity of the fluid and p the pressure. The spatial domain will be $\mathbb{R}^3$.

Henceforth, the following projected form of the equations will be useful:

$$\begin{aligned} &u_t - \triangle u + P\,\nabla \cdot (u \otimes u) = 0 \\ &\nabla \cdot u = 0 \end{aligned} \tag{4}$$

(P denotes the Leray projector). Let us mention here that if a distributional solution u to (4) on a space-time domain $\mathbb{R}^3 \times (0,T)$ is in $L^2((0,\tau), E)$ for all $0 < \tau < T$ where E denotes the closure of test functions in $L^2_{loc,unif}$, then there exists a p in $\mathcal{D}'\left(\mathbb{R}^3 \times (0,T)\right)$ such that

$$P\,\nabla \cdot (u \otimes u) = \nabla \cdot (u \otimes u) + \nabla p$$

in the sense of distributions, i.e. (u,p) is a distributional solution to (3). (The norm in $L^2_{loc,unif}$ is given by $\|v\|^2_{L^2_{loc,unif}} = \sup_{x_0 \in \mathbb{R}^3} \int_{|x-x_0|\le 1} |v|^2\,dx$.)

Next, we recall a definition of *suitable weak solutions.* Let (u,p) be a weak solution of the NSE on a space-time domain $\mathbb{R}^3 \times (0,T)$. Then, (u,p) is a suitable weak solution on $\mathbb{R}^3 \times (0,T)$ if

i) $\sup_{a<t<b} \int_{B(x_0,r_0)} |u|^2\,dx < \infty$

ii) $\iint_Q |\nabla \otimes u|^2\,dx\,dt < \infty$

iii) $\iint_Q |p|^{3/2}\,dx\,dt < \infty$

iv) $2\iint_Q |\nabla \otimes u|^2 \varphi\,dx\,dt \le \iint_Q |u|^2(\partial_t + \triangle)\varphi\,dx\,dt + \iint_Q (|u|^2 + 2p)(u \cdot \nabla)\varphi\,dx\,dt$

for all nonnegative $\varphi \in \mathcal{D}(Q)$ where $Q = B(x_0,r_0) \times (a,b)$ is an arbitrary space-time cylinder contained in $\mathbb{R}^3 \times (0,T)$.

Lemarié-Rieusset defined a locally uniform version of weak solutions, the so called *local Leray solutions.* The definition is as follows [L-R2]. Let $u_0 \in L^2_{loc,unif}$ with $\nabla \cdot u_0 = 0$. A local Leray solution u corresponding to the initial data u_0 is a locally square-integrable distributional solution of (3) on $\mathbb{R}^3 \times (0,T)$ with the following properties:

i) $u \in L^\infty\left((0,t), L^2_{loc,unif}\right)$ for all $t < T$

ii) $\sup_{x_0 \in \mathbb{R}^3} \iint_{0<s<t, |x-x_0|\le 1} |\nabla \otimes u|^2\,dx\,ds < \infty$ for all $t < T$

iii) $\lim_{t\to 0^+}\int_K |u-u_0|^2\,dx = 0$ for any compact subset K of $\mathbb{R}^3$
iv) u is a suitable weak solution.

The following result can be found in [L-R2] – the idea of the proof is to use the localized energy inequality as the main tool in constructing solutions originating from the $L^2_{loc,unif}$-initial data.

Theorem 2. [L-R2]

i): *Let $u_0 \in L^2_{loc,unif}$ be divergence free. Then, there exists a local Leray solution u on $\mathbb{R}^3 \times (0,T)$ for some $T > 0$.*

ii): *If, in addition, $u_0 \in E$, then $u \in L^\infty((0,t),E)$ for all $t < T$ and $\lim_{t\to 0+}\|u(\cdot,t)-u_0\|_{L^2_{loc,unif}} = 0$.*

The construction is based on deriving some (uniform in ϵ) locally uniform energy-type estimates on solutions to the family of mollified Navier-Stokes equations

$$(u_\epsilon)_t - \triangle u_\epsilon + P\,\nabla\cdot((u_\epsilon * \rho_\epsilon)\otimes u_\epsilon) = 0 \qquad (5)$$
$$\nabla\cdot u_\epsilon = 0$$

supplemented by $u_\epsilon(\cdot,0) = u_0$ for all $\epsilon > 0$. A family of (standard) mollifiers $\{\rho_\epsilon\}_{\epsilon>0}$ is defined as follows: let ρ be a nonnegative test function normalized such that $\int \rho\,dx = 1$–then ρ_ϵ is given by $\rho_\epsilon(x) = \frac{1}{\epsilon^3}\rho(\frac{x}{\epsilon})$ for all x in $\mathbb{R}^3$. (It was subsequently shown in [L-R2] that any local-in-time local Leray solution originating from E-inital data can be extended to a global-in-time local Leray solution–the proof is based on splitting of the initial data and a generalized Von Wahl uniqueness theorem.)

Next, we recall a definition of (space-time) singular/regular points of a weak solution u in the CKN sense. A point (x,t) in $\mathbb{R}^3\times(0,T)$ is *singular* if $\|u\|_{L^\infty(D)} = \infty$ for any space-time neighborhood D of (x,t). A point is *regular* if it is not singular, i.e. if there exists a space-time neighborhood D such that $\|u\|_{L^\infty(D)} < \infty$. The main partial regularity result is given by the following theorem.

Theorem 3. [CKN] *Let u be a suitable weak solution on $\mathbb{R}^3 \times (0,T)$. Then the one-dimensional Hausdorff measure of a possible singular set in $\mathbb{R}^3 \times (0,T)$ is 0. In particular, the singular set can not contain a smooth space-time curve.*

Remark 4. The original statement is somewhat stronger – it states that the one-dimensional parabolic Hausdorff measure of a singular set is 0.

At first, it may be puzzling that local boundedness is in this context identified as "regularity". However, the following result due to Serrin [S] implies smoothness in x at any regular point (x,t).

Theorem 5. [S] *Let u be a weak solution in some open space-time region D such that $u \in L^p_x L^q_t(D)$ for a pair of exponents (p,q) satisfying the Foias-Prodi-Serrin condition $\frac{3}{p}+\frac{2}{q} < 1$. Then u is of class C^∞ in x, and each derivative is bounded on compact subregions of D.*

Recall that a weak Lebesgue space $L^{p,\infty}$ $(p \ge 1)$ consists of all measurable functions u satisfying

$$\mu\big(\{x\in\mathbb{R}^3 : |u(x)| > s\}\big) \le \frac{c}{s^p}$$

for all $s > 0$, and an absolute constant $c > 0$, where μ denotes 3D Lebesgue measure. Note that homogeneous of degree -1 functions can live in the critical weak Lebesgue space $L^{3,\infty}$. Since the weak Lebesgue spaces are not separable, the linear heat semigroup $S(\cdot)$ is not strongly continuous at $t = 0$, and so one is bound to consider $C_w([0,T), L^{3,\infty})$, a space consisting of functions u satisfying

$$u \in L^\infty\big((0,T), L^{3,\infty}\big) \cap C\big((0,T), L^{3,\infty}\big) \tag{6}$$

$$u(\cdot,t) \to u(\cdot,0) \text{ weakly as } t \to 0^+. \tag{7}$$

Somewhat surprisingly, the NSE bilinear form $B = B(u,v) = -\int_0^t S(t-s)P\,\nabla \cdot (u \otimes v)(s)\,ds$ has better continuity properties in the $L^{3,\infty}$-based spaces than in the L^3-based spaces. It was shown in [O] that B is not continuous in $C([0,T), L^3)$. In contrast (cf., e.g., [M]), B is continuous in $X_T \equiv C_w([0,T), L^{3,\infty})$ for any $0 < T \leq \infty$. This fact yields, via a standard fixed-point argument, existence and uniqueness of small mild solutions in X_T, and hence existence of self-similar solutions (in X_∞) originating from the homogeneous of degree -1 initial data with small $L^{3,\infty}$-norm [M].

3. **Regularity of forward self-similar suitable weak solutions.** In this section, we observe that in the universe of suitable weak solutions, i.e., in the setting of the CKN theory, any forward-in-time self-similar solution is necessarily infinitely smooth in both space and time variables.

Theorem 6. *Let (u,p) be a self-similar suitable weak solution on $\mathbb{R}^3 \times (0,\infty)$. Then both u and p are infinitely smooth.*

Proof. First, we show that a possible singular set of u on $\mathbb{R}^3 \times (0,\infty)$ is empty. In what follows, it will be convenient to define a family of parabolic space-time cylinders centered at a space-time point (x,t). For $r > 0$, define $C_{x,t}(r)$ by

$$C_{x,t}(r) = \{(y,\tau) \in \mathbb{R}^3 \times (0,\infty) |\ |\tau - t| < r^2, |x-y| < r\}.$$

We argue by contradiction. Suppose there exists a singular point (x,t) of u in $\mathbb{R}^3 \times (0,\infty)$, and consider a family of parabolic cylinders $\{C_{x,t}(r)\}$ for $0 < r < \sqrt{t}$. Since (x,t) is a singular point, $\|u\|_{L^\infty\left(C_{x,t}(r)\right)} = \infty$ for all r. Fix any $\lambda > 0$, and then fix an r, $0 < r < \sqrt{t}$. Then, via the self-similarity of u,

$$\|u\|_{L^\infty\left(C_{\lambda x, \lambda^2 t}(\lambda r)\right)} = \lambda^{-1}\|u\|_{L^\infty\left(C_{x,t}(r)\right)} = \infty$$

for any r, $0 < r < \sqrt{t}$. In other words, we constructed a family of parabolic neighborhoods of a point $(\lambda x, \lambda^2 t)$ on which u blows up, and so $(\lambda x, \lambda^2 t)$ is also a singular point. The argument can be repeated for any $\lambda > 0$, and hence we generated a smooth space-time curve of singular points passing through the point (x,t) contained in $\mathbb{R}^3 \times (0,\infty)$. This contradicts Theorem 3 and thus the singular set must be empty.

Now, it is standard to infer infinite spatial regularity. Fix any space-time point (x,t). Since (x,t) is a regular point, there exists a bounded neighborhood D of (x,t) such that $\|u\|_{L^\infty(D)} < \infty$. This implies that $u \in L^p_x L^q_t(D)$ for any $1 \leq p,q, \leq \infty$, and in particular, u satisfies the condition of Theorem 5. It follows that $u \in C^\infty_x(D)$, and that each derivative is bounded on compact subregions of D.

Infinite time regularity in general does not follow from emptiness of the singular set. However, self-similarity will help. More precisely, we write $u(x,t) = \dfrac{1}{\sqrt{t}} U\left(\dfrac{x}{\sqrt{t}}\right)$ where $U(\cdot) = u(\cdot,1)$ for all (x,t) in $\mathbb{R}^3 \times (0,\infty)$. Since we already know that $u \in C^\infty_x$, and $u \in C^\infty_x$ if and only if $U \in C^\infty$, it follows that U is infinitely smooth. The infinite time regularity is now readily seen. □

4. **Regularity of Meyer's $L^{3,\infty}$ self-similar solutions.** Here, we give a proof that potentially singular, small, self-similar solutions constructed by Meyer in $X_\infty = C_w([0,\infty), L^{3,\infty})$ are actually regular.

Recall that the construction is based on continuity of the NSE bilinear form B in X_T for any $0 < T \leq \infty$. More precisely, it was shown in [M] that there exists a constant $C_M > 0$ such that $\|B(u,v)\|_{X_T} \leq C_M \|u\|_{X_T} \|v\|_{X_T}$ for all $u, v \in X_T$. One then follows a usual construction of mild solutions. Rewrite (4) in an integral form,

$$u(t) = S(t)u_0 + B(u,u)(t), \tag{8}$$

and note that if $\|S(\cdot)u_0\|_{X_T} < \dfrac{1}{4C_M}$, then via a standard fixed point argument, there exists a unique solution u in the ball of the radius $2\|S(\cdot)u_0\|_{X_T}$ in X_T. The smallness condition needed can be viewed as a smallness condition on the $L^{3,\infty}$ norm of u_0, since there exists an absolute constant $c > 0$ such that $\|S(t)u_0\|_{L^{3,\infty}} \leq c\|u_0\|_{L^{3,\infty}}$ for any $t > 0$, and all u_0 in $L^{3,\infty}$.

Uniqueness implies that, if in addition, initial data u_0 are homogeneous of degree -1, the solution u constructed via a fixed point algorithm is self-similar.

Theorem 7. *Any self-similar solution u resulting from the Meyer's construction in $C_w([0,\infty), L^{3,\infty})$ is necessarily in $C_x^\infty C_t^\infty\big(\mathbb{R}^3 \times (0,\infty)\big)$.*

Proof. We will see that u coincides with a local Leray solution u^* constructed in [L-R2], and since local Leray solutions are suitable, Theorem 6 can be applied to infer infinite smoothness in both space and time variables. This is simply a manifestation of 'weak=strong' principle (actually, 'weak=mild' here). A general statement can be found, e.g., in Theorem 13.4 [L-R2].

First, rewrite the equations for the family of smooth approximations $\{u_\epsilon\}_{\epsilon>0}$ (5) as

$$u_\epsilon(t) = S(t)u_0 + B(u_\epsilon * \rho_\epsilon, u_\epsilon)(t). \tag{9}$$

Let $0 < T \leq \infty$. Note that since $\|u * z\|_{X_T} \leq \|u\|_{X_T}\|z\|_1$,

$$\|B(u_\epsilon * \rho_\epsilon, u_\epsilon)\|_{X_T} \leq C_M \|u_\epsilon * \rho_\epsilon\|_{X_T} \|u_\epsilon\|_{X_T} \leq C_M \|u_\epsilon\|_{X_T}^2$$

($\|\rho_\epsilon\|_1 = 1$ for any $\epsilon > 0$), and hence we get exactly the same estimate as on u. Thus, for any $\epsilon > 0$, there exists a unique solution u_ϵ to (9) satisfying $\|u_\epsilon\|_{X_T} \leq 2\|S(\cdot)u_0\|_{X_T}$ (for any $0 < T \leq \infty$).

Now, we go back to the construction of local-in-time local Leray solutions in [L-R2] based on locally uniform (in space) energy estimates on the family $\{u_\epsilon\}$ obtained via the localized energy inequality, and starting from initial data in E– the closure of test functions in $L^2_{loc,unif}$. Note that since $L^{3,\infty} \hookrightarrow E$, our initial data u_0 belong to E as well. A solution u^* is constructed via a compactness argument. Recall that one of the convergences obtained in the proof of Theorem 2 is a strong convergence of a sequence $\{\varphi u_{\epsilon_n}\}$ to φu^* in $L^p\big((0,T), L^2\big)$ for any test function $\varphi \in \mathcal{D}(\mathbb{R}^3 \times (0,T))$, and any $p < \infty$ (for some $T > 0$). This implies the *weak**-convergence of $\{u_{\epsilon_n}\}$ to u^* in X_T–consequently, $u^* \in X_T$ and $\|u^*\|_{X_T} \leq \liminf_{n\to\infty} \|u_{\epsilon_n}\|_{X_T} \leq 2\|S(\cdot)u_0\|_{X_T}$. Thus $u^* = u$ by uniqueness of small solutions in X_T.

This implies that u is suitable on $\mathbb{R}^3 \times (0,T)$–hence on the whole space-time by self-similarity finishing the proof. □

Acknowledgments. The author thanks an anonymous referee for helpful comments.

REFERENCES

[B1] O. Barazza, *Self-similar solutions in weak L^p spaces of the Navier-Stokes equations*, Rev. Mat. Iberoam. **12** (1996), 411-439

[B2] O. Barazza, *Regularity and stability for the solutions of the Navier-Stokes equations in Lorentz spaces*, Nonlinear Anal. **35** (1999), 747-764

[CKN] L. Caffarelli, R. Kohn and L. Nirenberg, *Partial regularity of suitable weak solutions of the Navier-Stokes equations*, Comm. Pure Appl. Math. **35** (1982), 771-837

[C] M. Cannone, *Ondelettes, Paraproduits et Navier-Stokes*, Diderot Editeur, Paris, 1995

[CK] M. Cannone and G. Karch, *Smooth or singular solutions to the Navier-Stokes system ?*, J. Diff. Eq. **197** (2004), 247-274

[CMP] M. Cannone, Y. Meyer and F. Planchon, *Solutions auto-similaires des équations de Navier-Stokes dans* $\mathbb{R}^3$, Exposé n. VIII, Séminaire X-EDP, Ecole Polytechnique, 1994

[CP] M. Cannone and F. Planchon, *Self-similar solutions of the Navier-Stokes equations in* $\mathbb{R}^3$, Comm. PDE **21** (1996), 179-193

[GM] Y. Giga and T. Miyakawa, *Navier-Stokes flows in* $\mathbb{R}^3$ *with measures as initial vorticity and the Morrey spaces*, Comm. PDE **14** (1989), 577-618

[KY] H. Kozono and M. Yamazaki, *Local and global unique solvability of the Navier-Stokes exterior problem with Cauchy data in the space* $L^{n,\infty}$, Houston J. Math. **21** (1995), 755-799

[L-R1] P.G. Lemarié - Rieusset, *Solutions faibles d'énergie infinie pour les équations de Navier-Stokes dans* $\mathbb{R}^3$, C.R. Acad. Sci. Paris Sér. I **328 (8)** (1999), 1133-1138

[L-R2] P.G. Lemarié - Rieusset, *Recent developments in the Navier-Stokes problem*, Research Notes in Mathematics, Chapman & Hall / CRC, 2002

[L] J. Leray, *Sur le mouvement d'un liquide visqueaux emplissant l'espace*, Acta Math. **63** (1934), 193-248

[M] Y. Meyer, Wavelets, paraproducts and Navier-Stokes equations, in *Current Developments in Mathematics 1996*, International Press, 105-212 (1999)

[MNPS] J. Málek, J. Nečas, M. Pokorný and M.E. Schonbek, *On possible singular solutions to the Navier-Stokes equations*, Math. Nachr. **199** (1999), 97-114

[NRS] J. Nečas, M. Růžička and V. Šverák, *On Leray self-similar solutions of the Navier-Stokes equations*, Acta Math. **176** (1996), 283-294

[O] F. Oru, *Rôle des Oscillations dans Quelques Probleèmes d'Analyse Non Linéaire*, Doctoral Thesis, Ecole Normale Supériore de Cachan, France (1998)

[P1] F. Planchon, *Convergence de solutions des équations de Navier-Stokes vers de solutions auti-similaires*, Exposé n. III, Séminaire X-EDP, Ecole Polytechnique, 1996

[P2] F. Planchon, *Asymptotic behavior of global solutions to the Navier-Stokes equations in* $\mathbb{R}^3$, Rev. Mat. Iberoamericana **14 1** (1998), 71-93

[S] J. Serrin, *On the interior regularity of weak solutions of the Navier-Stokes equations*, Arch. Rat. Mech. Anal. **9** (1962), 187-195

[Ts] T.P. Tsai, *On Leray's self-similar solutions of the Navier-Stokes equations satisfying local energy estimates*, Arch. Rational Mech. Anal. **143** (1998), 29-51

[Y] M. Yamazaki, *The Navier-Stokes equations in the weak-L^n spaces with time-dependent external force*, Math. Ann. **317** (2000), 635-675.

[W] J. Wu, *Regularity results for weak solutions of the 3D MHD equations*, Discrete Cont. Dynamical Systems **10** (2004), 543-556.

Received January 2005; revised October 2005.

E-mail address: zg7c@virginia.edu