# Vortex stretching and anisotropic diffusion in the 3D Navier-Stokes equations

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*Dedicated to Professor Hugo Beirão da Veiga on the occasion of his 70th birthday, with admiration*

ABSTRACT. The goal of this article is to present – in a cohesive, and somewhat self-contained fashion – several recent results revealing an experimentally, numerically, and mathematical analysis-supported *geometric scenario* manifesting *large data* logarithmic *sub-criticality* of the 3D Navier-Stokes regularity problem. Shortly – in this scenario – the *transversal small scales* produced by the mechanism of vortex stretching (coupled with the decay of the volume of the regions of intense vorticity) reach the threshold sufficient for the *locally anisotropic diffusion* to engage and control the sup-norm of the vorticity, preventing the (possible) formation of finite time singularities.

## 1. Prologue

Vortex stretching has been viewed as the principal *physical mechanism* responsible for the vigorous creation of *small scales* in turbulent fluid flows. This goes back at least to G. I. Taylor's fundamental paper "Production and dissipation of vorticity in a turbulent fluid" from 1937 [**Tay37**].

While the *production* part has been relatively well-understood (the amplification of the vorticity via the process of vortex stretching follows essentially from the conservation of the angular momentum in the incompressible fluid), the precise physics/mathematics behind the vortex stretching-induced *dissipation* is less transparent. For his part, Taylor inferred the thoughts on the *anisotropic dissipation* chiefly from the wind tunnel *measurements* of turbulent flow past a uniform grid, concluding the paper with the following sentence.

"It seems that the stretching of vortex filaments must be regarded as the principal mechanical cause of the high rate of dissipation which is associated with turbulent motion."

Since then, it has been a grand challenge in the mathematical fluid mechanics community to try to explain/quantify the process of anisotropic dissipation in turbulent flows directly from the mathematical model – the 3D Navier-Stokes equations (NSE).

Numerical simulations (cf. [**AKKG87**,**JWSR93**,**SJO91**,**VM94**]) reveal that the regions of intense vorticity are dominated by *coherent vortex structures* and in particular, *vortex filaments*. There are two imminent morphological signatures of this geometry. One is *local coherence of the vorticity direction*, and the other one is *local existence of sparse/thin direction(s)*.

`Local coherence.  Geometric depletion of the nonlinearity.`

The pioneering work in this direction was presented by Constantin in [**Co94**] where he derived a singular integral representation of the stretching factor in the evolution of the vorticity magnitude featuring a geometric kernel that is depleted by *local coherence* of the *vorticity direction*, a purely geometric condition. This has been referred to as *geometric depletion of the nonlinearity*, and has led to the first rigorous confirmation of the phenomenon of anisotropic dissipation utilizing the 3D NSE, a theorem ([**CoFe93**]) stating that as long as the vorticity direction is Lipschitz-coherent (in the regions of high vorticity), the $L^2$-norm of the vorticity is controlled, and no finite time blow-up can occur. The Lipschitz-coherence condition was later scaled down to $\frac{1}{2}$-Hölder in [**daVeigaBe02**], and a full *spatiotemporal localization* of the $\frac{1}{2}$-Hölder condition was performed in [**Gr09**] (a different approach to localization was previously introduced in [**ChKaLe07**]). A family of local, hybrid geometric-analytic regularity criteria including a *scaling invariant* improvement of the $\frac{1}{2}$-Hölder condition was presented in [**GrGu10-1**]. The study of the coherence of the vorticity direction up to the boundary-regularity criteria in the case of the no-stress boundary conditions was presented in [**daVeigaBe09**], and in the case of the no-slip boundary conditions in [**daVeiga07**].

Essentially, an unhappy event preventing the manifestation of the anisotropic dissipation in this setting is the one of 'crossing of the vortex lines', i.e., of the vorticity direction forming even a simple spatial discontinuity – two different limit points – at a (possible) singular time.

`Local anisotropic sparseness. Vortex stretching-anisotropic diffusion.`

An alternative mathematical description of the anisotropic dissipation in the 3D incompressible viscous flows was recently exposed in [**Gr13**], and is based on the concept of *anisotropic diffusion*. Taken at face value, the 3D NSE diffusion – generated by the Laplacian – is isotropic. The (isotropic) diffusion is then utilized via sharp, local-in-time *spatial analyticity* properties of solutions in $L^\infty$ which provide an ambient amenable to the application of the *harmonic measure majorization principle*. More precisely, as long as the region of intense vorticity, defined to be the region in which the vorticity magnitude exceeds a fraction of the $L^\infty$ norm, exhibits the property of *local existence of a sparse/thin direction* at a scale comparable to the radius of spatial analyticity (essentially, $\frac{1}{C}\|\omega(t)\|_\infty^{-\frac{1}{2}}$ where $\omega$ denotes the vorticity of the fluid), an argument relying on the translational and rotational invariance of the equations and certain geometric properties of the harmonic measure (this is what introduces *anisotropy*), and the harmonic measure maximum principle, shows that the $L^\infty$ norm of the vorticity is controlled, and no finite-time blow up can occur. It is worth mentioning that it suffices to assume the aforementioned sparseness property *intermittently in time*.

Of course, a key question is whether there is any evidence, either numerical, or mathematical, that the *scale of local linear sparseness/thinness* needed for triggering the mechanism of anisotropic diffusion is in fact achieved in a turbulent flow. Thinking in terms of vortex filaments, the scale we are interested in is essentially the *length scale of the diameters of the cross-sections*, i.e., the *transversal scale* of the filament. It appears easier – both numerically and analytically – to estimate the *axial length scale of the filaments* instead. This, coupled with a suitable estimate on the volume of the region of the intense vorticity, provides an (indirect) estimate on the desired length scale.

Direct numerical simulations suggest that – intermittently in time/in the time average – the axial lengths of the filaments are essentially comparable to the *macro scale* (e.g., the side length $L$ in the case of the $L$-periodic boundary conditions). On the other hand, the *a priori* $L^1$-estimate on the vorticity [**Co90**] implies that the volume of the region of intense vorticity is bounded by $C\|\omega(t)\|_\infty^{-1}$. Hence (intermittently in time), the transversal *micro scale* of the filament is bounded by $C\|\omega(t)\|_\infty^{-\frac{1}{2}}$; in other words, the NSE regularity problem in this scenario becomes *critical*.

In addition to the numerical evidence, a very recent work [**DaGr12-3**] presented a *mathematical evidence* of creation and persistence (in the time average) of the macro scale-long vortex filaments. More precisely, exploiting a *dynamic, spatial multi-scale ensemble averaging process* designed to detect *sign-fluctuations* of an *a priori* sign varying physical quantity across scales, it was shown that there exists a range of scales – extending from a suitable micro scale to the macro scale – at which the vortex stretching term is essentially positive. (The averaging procedure utilized had been previously developed in a recent series of papers [**DaGr11-1**, **DaGr11-2**, **DaGr12-1**, **DaGr12-2**] as a mathematical framework for the study of turbulent cascades in *physical scales* of 3D incompressible fluid flows.)

The aforementioned ruminations offer a physically, numerically, and mathematical analysis-supported *large data criticality scenario* for the 3D NSE. The NSE themselves are (still) *super-critical*; regardless of the functional setup, there has been a 'scaling gap' between a regularity criterion in view and the corresponding *a priori* bound. An instructive example is given by the regularity condition $u \in L_t^\infty L_x^3$ obtained by Escauriaza, Seregin and Sverak in [**ESS03**], to be contrasted to Leray's *a priori* bound $u \in L_t^\infty L_x^2$ ([**Le34**]).

A natural question to ask is whether it is possible to *break the criticality* in this setting; i.e., whether the intricate interplay between the vortex stretching and the anisotropic diffusion results in preventing the formation of singularities, rather than in a critical blow-up scenario. In a very recent article [**BrGr13-2**], it was shown that a *very mild, purely geometric* assumption yields a uniform-in-time $L \log L$ bound on the vorticity; this in turn implies an extra-log decay of the vorticity distribution function, i.e., of the volume of the region of intense vorticity, breaking the scaling, and transforming the aforementioned criticality scenario into an *anisotropic diffusion-win scenario* (no singularities). More precisely, the assumption is a uniform-in-time boundedness of the localized *vorticity direction* in a suitable, logarithmically weighted, local space of *bounded mean oscillations* ($BMO$). An interesting feature of this space is that it allows for discontinuous functions exhibiting singularities of, e.g., $\sin \log |\log( \text{something algebraic})|$-type.

Hence, the vorticity direction can form a singularity in a geometrically spectacular fashion – every point on the unit sphere being a limit point – and the $L \log L$ bound will still hold (in particular, a simple 'crossing of the vortex lines' is not an obstruction).

The proof is based on an adaptation of the method utilized in [**Co90**], the novel components being exploiting *analytic cancelations* in the vortex-stretching term via a version of the Div-Curl Lemma (in the sense of Coifman, Lions, Meyer and Semmes theory of compensated compactness in Hardy spaces), a local version of the $\mathcal{H}^1 - BMO$ duality, a sharp pointwise multiplier theorem in local $BMO$ spaces, and Coifman-Rochberg's $BMO$-estimate on the logarithm of the *maximal function* of a locally integrable function (the estimate is *independent of the function* and depending only on the dimension of the space).

This result ([**BrGr13-2**]) is – in a way – complementary to the results obtained in [**BrGr13-1**]. The class of conditions leading to an $L \log L$-bound presented in [**BrGr13-1**] consists of suitable *blow-up* rates that can be characterized as 'wild in time' with a uniform spatial (e.g., algebraic) structure, while the condition presented in [**BrGr13-2**] can be characterized as 'wild in space' and uniform in time.

In summary, the papers [**Gr13**, **DaGr12-3**, **BrGr13-2**] can be viewed as providing a rigorous mathematical framework (directly from the 3D NSE) for justification of Taylor's view on vortex stretching as the principal mechanical cause for the high rate of dissipation in turbulent flows. Incidentally, they also point to a possible new direction in the study of the 3D NSE regularity problem.

## 2. Anisotropic diffusion

3D Navier-Stokes equations (NSE) – describing a flow of 3D incompressible viscous fluid – read

$$u_t + (u \cdot \nabla)u = -\nabla p + \nu \triangle u,$$

supplemented with the incompressibility condition $\operatorname{div} u = 0$, where $u$ is the velocity of the fluid, $p$ is the pressure, and $\nu$ is the viscosity. Taking the curl yields the vorticity formulation,

$$\omega_t + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u + \nu \triangle \omega,$$

where $\omega = \operatorname{curl} u$ is the vorticity.

Computational simulations of *3D homogeneous turbulence* reveal that the regions of intense vorticity organize in *coherent vortex structures*, and in particular, in elongated vortex tubes/filaments, cf. [**S81**, **AKKG87**, **SJO91**, **JWSR93**, **VM94**]. An in-depth analysis of creation and dynamics of vortex tubes in 3D turbulent flows was presented in [**CPS95**]; in particular, a suitably defined *dynamical scale of coherence* of the vorticity direction field was estimated. The current body of work containing analytical, as well as analytical and numerical results on the dynamics of coherent vortex structures includes [**GGH97**, **GFD99**, **Oh09**, **Hou09**].

In what follows, we will focus on *sparseness*.

DEFINITION 2.1. Let $x_0$ be a point in $\mathbb{R}^3$, $r > 0$, $S$ an open subset of $\mathbb{R}^3$ and $\delta$ in $(0, 1)$. The set $S$ is *linearly $\delta$-sparse around $x_0$ at scale $r$ in weak sense* if there exists a unit vector $d$ in $S^2$ such that

$$\frac{|S \cap (x_0 - rd, x_0 + rd)|}{2r} \leq \delta.$$

Denote by $\Omega_t(M) = \{x \in \mathbb{R}^3 : |\omega(x, t)| > M\}$ the vorticity super-level set at time $t$. Then the following manifestation of anisotropic diffusion holds ([**Gr13**]).

THEOREM 2.1. *Suppose that a solution $u$ is regular on an interval $(0, T^*)$.*

*Assume that either*

*(i) there exists $t$ in $(0, T^*)$ such that $t + \dfrac{1}{d_0^2 \|\omega(t)\|_\infty} \geq T^*$, or*

*(ii) $t + \dfrac{1}{d_0^2 \|\omega(t)\|_\infty} < T^*$ for all $t$ in $(0, T^*)$, and there exists $\epsilon$ in $(0, T^*)$ such that for any $t$ in $(T^* - \epsilon, T^*)$, there exists $s = s(t)$ in $\left[t + \dfrac{1}{4d_0^2 \|\omega(t)\|_\infty}, t + \dfrac{1}{d_0^2 \|\omega(t)\|_\infty}\right]$ with the property that for any spatial point $x_0$, there exists a scale $r = r(x_0)$, $0 < r \leq \dfrac{1}{2d_0^2 \|\omega(t)\|_\infty^{\frac{1}{2}}}$, such that the super-level set $\Omega_s(M)$ is linearly $\delta$-sparse around $x_0$ at scale $r$ in weak sense; here, $\delta = \delta(x_0)$ is an arbitrary value in $(0, 1)$, $h = h(\delta) = \dfrac{2}{\pi} \arcsin \dfrac{1 - \delta^2}{1 + \delta^2}$, $\alpha = \alpha(\delta) \geq \dfrac{1 - h}{h}$, and $M = M(\delta) = \dfrac{1}{d_0^\alpha} \|\omega(t)\|_\infty$.*

*Then, there exists $\gamma > 0$ such that $\omega$ is in $L^\infty\left((T^* - \epsilon, T^* + \gamma); L^\infty\right)$, i.e., $T^*$ is not a singular time. ($d_0$ is a suitable absolute constant.)*

The quantity $(d_0^2 \|\omega(t)\|_\infty)^{-1}$ is the time step in the local-in-time well-posedness scheme in $L^\infty$ initiated at $t$. The scheme can be complexified [**GrKu98**, **Gu10**]. For any $s \in \left(t, t + (d_0^2 \|\omega(t)\|_\infty)^{-1}\right)$, $\omega(s)$ is a restriction of the function holomorphic in the region $\{x + iy \in \mathbb{C}^3 : |y| < 1/c_1\sqrt{s}\}$; moreover, the sup-norm of the complexified solution is controlled by $c_2 \|\omega(t)\|_\infty$.

The idea of the proof is as follows. Let $x_0$ be a spatial point, and $d = d(x_0)$ a sparse direction within the region of intense vorticity, at the scale comparable to the uniform lower bound on the radius of spatial analyticity. By the translational invariance of the equations, we can send $x_0$ to the origin, and by the rotational invariance, we can align $d$ with one of the coordinate directions. The (real) coordinate is then embedded in the complex plane, and the *harmonic measure maximum principle* applied with respect to the disk centered at the origin – with the radius comparable to the analyticity radius – is utilized to exploit the sparseness condition resulting in a "self-improving" bound on the sup-norm of the complexified vorticity, preventing the finite time blow-up.

The main engine behind the argument is local-in-time spatially analytic smoothing in $L^\infty$, a strong manifestation of the (isotropic) diffusion generated by $\partial_t - \triangle$; a *locally anisotropic diffusion effect* is a consequence of the translational and rotational invariance of the equations, and geometric properties of the harmonic measure.

REMARK 2.1. It suffices to assume the sparseness condition at (suitably chosen) *finitely many* times/intermittently in time.

### 3. A possible road to criticality

Adopting the notation introduced in the preceding section, define *the region of intense vorticity* to be the set $\Omega_{s(t)}\Big(\dfrac{1}{c_1}\|\omega(t)\|_\infty\Big)$ for an appropriate $c_1 > 1$. Let $R_0$ be a suitable *macro scale* associated with the flow. Computational simulations indicate that (intermittently-in-time) dominant geometry in the region of intense vorticity is the one of $R_0$-long vortex filaments; in order to estimate the transversal micro-scale of the filaments, it suffices to have a good estimate on the rate of the decrease of the volume of the vorticity super-level sets.

Let $(0, T)$ be an interval of interest. In [**Co90**], provided the initial vorticity is a bounded measure (and the initial velocity is of finite energy), Constantin showed that a corresponding weak solution satisfies $\displaystyle\sup_{t \in (0,T)} \|\omega(t)\|_{L^1} \le c_{0,T} = c(u_0, \omega_0, T)$. Chebyshev's inequality then implies

$$\text{Vol}\left(\Omega_{s(t)}\Big(\frac{1}{c_1}\|\omega(t)\|_\infty\Big)\right) \le \frac{c'_{0,T}}{\|\omega(t)\|_\infty} \quad (c'_{0,T} > 1),$$

which – in turn – yields the decrease of the transversal micro-scale of the filaments of at least at least $\dfrac{c''_{0,T}}{\|\omega(t)\|_\infty^{\frac{1}{2}}}$ $(c''_{0,T} > 1)$. This is precisely the scale of local, linear sparseness needed to trigger the mechanism of anisotropic diffusion exposed in the previous section, i.e., we arrive at *criticality*.

It is instructive to check the scaling in the *geometrically worst case scenario* – no sparseness – the super level set being clumped in a ball. In this case, the criticality would require $\lambda_{\omega(t)}(\beta) = O\Big(\dfrac{1}{\beta^{3/2}}\Big)$ uniformly in $(T^* - \epsilon, T^*)$ ($\lambda$ denotes the distribution function); this is a scaling-invariant condition – back to super-criticality of the problem, $O\Big(\dfrac{1}{\beta^{3/2}}\Big)$ *vs.* $O\Big(\dfrac{1}{\beta^1}\Big)$. (In fact, this is precisely the vorticity analogue of the velocity scaling gap – $L_t^\infty L_x^3$ *vs.* $L_t^\infty L_x^2$.)

Summarizing – in this scenario – the vortex stretching acts as the mechanism bridging (literally) the scaling gap in the regularity problem.

### 4. Mathematical evidence of criticality

In this section, we identify the range of scales of *positivity of the vortex-stretching term* $S\omega \cdot \omega$ ($S$ denotes the symmetric part of the gradient of $u$); this corresponds to the range of scales of creation and persistence of vortex filaments.

To this end, we exploit a spatial *multi-scale averaging method* designed to detect *sign fluctuations* of a quantity of interest across *physical scales* recently introduced in the study of turbulent transport rates in 3D incompressible fluid flows [**DaGr11-1**, **DaGr11-2**, **DaGr12-1**, **DaGr12-2**].

Let $B(0, R_0)$ be a macro-scale domain. A *physical scale* $R$, $0 < R \le R_0$, is realized via suitable ensemble averaging of the localized quantities with respect to '$(K_1, K_2)$-covers at scale $R$'.

DEFINITION 4.1. Let $K_1$ and $K_2$ be two positive integers; a cover $\{B(x_i, R)\}_{i=1}^{n}$ of $B(0, R_0)$ is a $(K_1, K_2)$-*cover at scale* $R$ if $\left(\dfrac{R_0}{R}\right)^3 \leq n \leq K_1 \left(\dfrac{R_0}{R}\right)^3$, and any point $x$ in $B(0, R_0)$ is covered by at most $K_2$ balls $B(x_i, 2R)$.

The parameters $K_1$ and $K_2$ are the maximal *global* and *local multiplicities*, respectively.

For a physical density of interest $f$, consider – localized to the cover elements $B(x_i, R)$ (per unit mass) – local quantities $\hat{f}_{x_i, R}$,

$$\hat{f}_{x_i, R} = \frac{1}{R^3} \int_{B(x_i, 2R)} f(x) \psi_{x_i, R}^{\delta}(x) \, dx,$$

for some $0 < \delta \leq 1$. The smooth cut-offs $\psi_i = \psi_{x_i, R}$ are equal to 1 on $B(x_i, R)$, vanish outside of $B(x_i, 2R)$, and satisfy $|\nabla \psi_i| \leq c_\rho \dfrac{1}{R} \psi_i^\rho, |\triangle \psi_i| \leq c_\rho \dfrac{1}{R^2} \psi_i^{2\rho-1}$, for a suitably chosen $\rho$, $\frac{1}{2} < \rho < 1$. Denote by $\psi_0$ the cut-off corresponding to the macro-scale domain $B(0, R_0)$. The cut-offs associated with 'boundary elements', i.e., the cover elements $B(x_i, R)$ intersecting the boundary of the macro-scale domain, are modified to satisfy certain compatibility relations with the global cut-off $\psi_0$; for technical details see, e.g., [**DaGr11-1**].

Denote by $\langle F \rangle_R$ the *ensemble average* given by $\langle F \rangle_R = \dfrac{1}{n} \sum_{i=1}^{n} \hat{f}_{x_i, R}$. The key feature of $\{\langle F \rangle_R\}_{0 < R \leq R_0}$ is that $\langle F \rangle_R$ being *stable* – i.e., nearly-independent on a particular choice of the cover (with the fixed local multiplicity $K_2$) – indicates there are *no significant sign fluctuations* at scales comparable or greater than $R$. On the other hand, if $f$ does exhibit significant sign fluctuations at scales comparable or greater than $R$, suitable *rearrangements* of the cover elements up to the maximal multiplicity – emphasizing first the positive and then the negative parts of $f$ – will result in $\langle F \rangle_R$ experiencing a wide range of values, from positive through zero to negative, respectively (the larger $K_2$, the finer detection).

For a non-negative density $f$, the ensemble averages are all comparable to each other throughout the full range of scales, $0 < R \leq R_0$; in particular, they are all comparable to the simple average over the macro-scale domain,

$$(4.1) \qquad\qquad \frac{1}{K_1} F_0 \leq \langle F \rangle_R \leq K_2 F_0,$$

for all $0 < R \leq R_0$, where $F_0 = \dfrac{1}{R_0^3} \int f(x) \psi_0^\delta(x) \, dx$.

Back to vortex stretching. Denote the time-averaged localized vortex-stretching terms per unit mass associated to the cover element $B(x_i, R)$ by $VST_{x_i, R, t}$,

$$(4.2) \qquad\qquad VST_{x_i, R, t} = \frac{1}{t} \int_0^t \frac{1}{R^3} \int (\omega \cdot \nabla) u \cdot \omega \; \phi_i \, dx \, ds;$$

here, $\phi_i = \phi_i(x, s) = \eta(s) \, \psi_{x_i, R}(x)$, where $\eta$ is a smooth function on $(0, T)$ satisfying $\eta = 0$ on $(0, 1/3\,T)$, $\eta = 1$ on $(2/3\,T, T)$, $|\eta'| \leq c_\kappa \frac{1}{T} \eta^\kappa$, for a suitable $\kappa$, $0 < \kappa < 1$.

The quantity of interest is the ensemble average of $\{VST_{x_i,R,t}\}_{i=1}^n$,

$$(4.3) \qquad \langle VST \rangle_{R,t} = \frac{1}{n} \sum_{i=1}^n VST_{x_i,R,t}.$$

$B(x_i, R)$-localized *enstrophy level dynamics* is then as follows,

$$
\begin{aligned}
\int_0^t \int (\omega \cdot \nabla) u \cdot \phi_i \, \omega \, dx \, ds = &\int \frac{1}{2} |\omega(x,t)|^2 \psi_i(x) \, dx + \int_0^t \int |\nabla \omega|^2 \phi_i \, dx \, ds \\
&- \int_0^t \int \frac{1}{2} |\omega|^2 \big( (\phi_i)_s + \triangle \phi_i \big) \, dx \, ds \\
(4.4) \qquad &- \int_0^t \int \frac{1}{2} |\omega|^2 (u \cdot \nabla \phi_i) \, dx \, ds,
\end{aligned}
$$

for any $t$ in $(2/3\,T, T)$, and $1 \le i \le n$. The hope is that the ensemble-averaging of the right-hand side detects a *dynamic range of scales* of positivity of the vortex stretching term $S\omega \cdot \omega$.

Before stating the result, several macro-scale quantities need to be introduced. Denote by $E_{0,t}$ time-averaged enstrophy per unit mass associated with the macro scale domain $B(0, 2R_0) \times (0,t)$,

$$
E_{0,t} = \frac{1}{t} \int_0^t \frac{1}{R_0^3} \int \frac{1}{2} |\omega|^2 \phi_0^{1/2} \, dx \, ds,
$$

by $P_{0,t}$ a modified time-averaged palinstrophy per unit mass,

$$
P_{0,t} = \frac{1}{t} \int_0^t \frac{1}{R_0^3} \int |\nabla \omega|^2 \phi_0 \, dx \, ds + \frac{1}{t} \frac{1}{R_0^3} \int \frac{1}{2} |\omega(x,t)|^2 \psi_0(x) \, dx
$$

(the modification is due to the shape of the temporal cut-off $\eta$), and by $\sigma_{0,t}$ a corresponding Kraichnan-type scale, $\sigma_{0,t} = \left( \dfrac{E_{0,t}}{P_{0,t}} \right)^{\frac{1}{2}}$. Then the following holds [**DaGr12-3**].

THEOREM 4.1. *Let $u$ be a global-in-time local Leray solution on $\mathbb{R}^3 \times (0, \infty)$, regular on $(0,T)$. Suppose that, for some $t \in (2/3\,T, T)$,*

$$(4.5) \qquad C \max\{M_0^{\frac{1}{2}}, R_0^{\frac{1}{2}}\} \, \sigma_{0,t}^{\frac{1}{2}} < R_0$$

*where $M_0 = \sup\limits_t \int_{B(0,2R_0)} |u|^2 < \infty$, and $C > 1$ a suitable constant depending only on the cover parameters.*
*Then,*

$$(4.6) \qquad \frac{1}{C} P_{0,t} \le \langle VST \rangle_{R,t} \le C P_{0,t}$$

*for all $R$ satisfying*

$$(4.7) \qquad C \max\{M_0^{\frac{1}{2}}, R_0^{\frac{1}{2}}\} \, \sigma_{0,t}^{\frac{1}{2}} \le R \le R_0.$$

A couple of remarks.

(i) Suppose that $T$ is the first (possible) singular time, and that the macro-scale domain contains some of the spatial singularities (at time $T$). This, paired with the *a priori* regularity on $u$ implies $\sigma_{0,t} \to 0$, $t \to T^-$; hence, the condition (4.5) in the theorem is *automatically satisfied* for any $t$ sufficiently close to the singular time $T$.

(ii) $P_{0,t} \to \infty$, $t \to T^-$, i.e., the vortex stretching intensifies as we approach the singularity.

(iii) The power of $\frac{1}{2}$ on $\sigma_{0,t}$ is a correction originating in the need for a suitable control of the localized transport term.

## 5. Logarithmic sub-criticality

The purpose of this section is to show how a *very mild, purely geometric,* condition transforms the criticality scenario exposed in the previous sections into a log sub-critical scenario, preventing the possible formation of singularities.

The idea is to get a uniform-in-time $L \log L$-bound on $w$, where $w = \sqrt{1 + |\omega|^2}$; this would impose an extra decay on the distribution function of the vorticity, breaking the criticality.

Suppose that the solution in view is smooth on $(0, T)$, and focus on some macro-scale spatial domain, e.g., $B(0, R_0)$. The evolution of $w$ satisfies the following partial differential inequality ([**Co90**]),

$$(5.1) \qquad \partial_t w - \triangle w + (u \cdot \nabla) w \leq \omega \cdot \nabla u \cdot \frac{\omega}{w}.$$

Since our goal is to control the evolution of $w \log w$ over $B(0, R_0)$, it is convenient to multiply (5.1) by $\psi (1 + \log w)$ where $\psi = \psi_0$ (a smooth cut-off associated with the macro-scale domain as in the previous section). After a fair amount of calculation, the following bound transpires,

$$I(\tau) \equiv \int \psi(x) \, w(x, \tau) \log w(x, \tau) \, dx \leq I(0) + c \int_0^\tau \int_x \omega \cdot \nabla u \cdot \psi \, \xi \, \log w \, dx \, dt$$
$$+ \ a \ priori \ \text{ bounded,}$$

for any $\tau$ in $[0, T)$; $\xi$ denotes the vorticity direction.

Note that with respect to the scaling of *a priori* bounded quantities, the integral on the right-hand side is log super-critical. The strategy to overcome this is as follows. Exploit the *analytic cancellations* in the vortex-stretching term $\omega \cdot \nabla u$ utilizing a version of the Div-Curl Lemma in the Hardy space $\mathcal{H}^1$, and then transform the gain via $\mathcal{H}^1 - BMO$ duality into some sort of an oscillation condition on $\xi$. To do this in an efficient manner, we will need a sharp pointwise multiplier theorem in a version of local $BMO$, and a result quantifying the intimate relationship between log and $BMO$; namely, the Coifman-Rochberg's estimate. For more details, as well as the references, see [**BrGr13-2**].

Recall that the space of bounded mean oscillations, $BMO$, is defined as follows

$$BMO = \left\{ f \in L^1_{loc} : \sup_{x \in \mathbb{R}^n, r > 0} \Omega\big(f, I(x, r)\big) < \infty \right\}$$

where $\Omega\big(f, I(x,r)\big) = \frac{1}{|I(x,r)|} \int_{I(x,r)} |f(x) - f_I| \, dx$ is the mean oscillation of the function $f$ with respect to its mean $f_I = \frac{1}{|I(x,r)|} \int_{I(x,r)} f(x) \, dx$, over the cube $I(x,r)$ centered at $x$ with the side-length $r$.

When $f \in L^1$, we can focus on small scales, e.g., $0 < r < \frac{1}{2}$. Let $\phi$ be a positive, non-decreasing function on $(0, \frac{1}{2})$, and consider the following version of local weighted spaces of bounded mean oscillations,

$$\|f\|_{\widetilde{bmo}_\phi} = \|f\|_{L^1} + \sup_{x \in \mathbb{R}^n, 0 < r < \frac{1}{2}} \frac{\Omega\big(f, I(x,r)\big)}{\phi(r)}.$$

Of special interest will be the space $\widetilde{bmo}_{\frac{1}{|\log r|}}$.

The following result can be found in [**BrGr13-2**].

THEOREM 5.1. *Let $u$ be a Leray solution to the 3D NSE. Assume that the initial vorticity $\omega_0$ is in $L^1 \cap L^2$, and that $T > 0$ is the first (possible) blow-up time. Suppose that*

$$\sup_{t \in (0,T)} \|(\psi\xi)(\cdot, t)\|_{\widetilde{bmo}_{\frac{1}{|\log r|}}} < \infty.$$

*Then,*

$$\sup_{t \in (0,T)} \int \psi(x) \, w(x,t) \log w(x,t) \, dx < \infty.$$

`Good news.` $\widetilde{bmo}_\phi$ contains discontinuous functions if and only if $\int_0^{\frac{1}{2}} \frac{\phi(r)}{r} \, dr = \infty$. More specifically, $\widetilde{bmo}_{\frac{1}{|\log r|}}$ contains bounded functions with the discontinuities of, say, $\sin \log |\log(\text{something algebraic})|$-type, i.e., $\xi$ can (as it approaches $T$, and the spatial singularity at $T$) oscillate among *infinitely many limit points* on the unit sphere, and still yield extra-log decay of the distribution function of $\omega$ breaking the criticality.

## 6. Epilogue

From the fluid mechanics perspective, the results reviewed provide a framework for *rigorous identification* of the interplay between vortex stretching and anisotropic diffusion as a principal mechanism behind the phenomenon of *turbulent dissipation*.

From the PDE perspective, they identify a *large data geometric sub-criticality scenario* in the 3D NS regularity problem. This is achieved in two steps. First, a *dynamic criticality scenario* is revealed – thinking in terms of vortex filaments – in which the transversal scale of the filaments matches the scale of local, linear (anisotropic) sparseness of the region of intense vorticity needed to trigger the anisotropic diffusion [**Gr13, DaGr12-3**]. Then, a very mild geometric condition – boundedness of the vorticity direction in space $\widetilde{bmo}_{\frac{1}{|\log r|}}$ – *breaking the criticality* is identified. In particular, the vorticity direction is allowed to *develop spatial discontinuities* at the possible singular time $T$ [**BrGr13-2**]. It is instructive to briefly compare this to several (relatively) recent results from the literature in which a form of *criticality* is *assumed*, and then an *anisotropic condition* implying the regularity is identified. In [**SeSv09**] (see also [**CSYT08, CSTY09**]), it is shown

– under the type I blow-up assumption – that the local *axisymmetric solutions* do not form singularities. The regularity condition here is the one of *global anisotropy*. In [**GiMi11**], the authors showed – also under the type I blow-up assumption – that as long as the vorticity direction possesses a uniform *modulus of continuity*, no finite time blow-up can occur. As in [**BrGr13-2**], the regularity condition here is the one of *local anisotropy*; however, in contrast to [**BrGr13-2**], uniform continuity of the vorticity direction is still required.

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