

Energy Cascades and Flux Locality in Physical Scales of the 3D Navier-Stokes Equations

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Abstract: Rigorous estimates for the total – (kinetic) energy plus pressure – flux in \mathbb{R}^3 are obtained from the three dimensional Navier-Stokes equations. The bounds are used to establish a condition – involving Taylor length scale and the size of the domain – sufficient for existence of the inertial range and the energy cascade in decaying turbulence (zero driving force, non-increasing global energy). Several manifestations of the locality of the flux under this condition are obtained. All the scales involved are *actual physical scales* in \mathbb{R}^3 and no regularity or homogeneity/scaling assumptions are made.

1. Introduction

One of the main features of Kolmogorov's empirical turbulence theory [10–12] is existence of *energy cascade* over a wide range of length scales, called *the inertial range*, where the dissipation effects are dominated by the transport of energy from higher to lower scales. Energy cascades have been observed in physical experiments, but theoretical justification of this phenomenon using equations of fluid motion, and in particular, the Navier-Stokes equations (NSE), remains far from being settled. The technical complexity of the NSE makes it difficult to establish conditions under which such cascades can occur. A particular problem is the possible lack of regularity of the solutions to the NSE, and thus choosing the right setting becomes crucial. (For an overview of various mathematical models of turbulence and the theory of the NSE, see, e.g., [6,7,9] and [4,14,18], respectively.)

The first studies in this direction were made in [8], where infinite-time averages of the Leray-Hopf solutions in the Fourier setting were used to establish a sufficient condition for the energy cascade. This condition, involving Taylor length scale, provided an inspiration for the sufficient condition (4.13) obtained in Sect. 4. In contrast to [8], our goal was to work in *physical space*, dealing with actual length scales in \mathbb{R}^3 rather than the Fourier wave numbers.

In studying a PDE model, a natural way of introducing a concept of scale is to measure oscillations, i.e., (distributional) derivatives of a quantity with respect to the scale.

Considering an L^1_{loc} function f on a ball of radius $2R$, $B(\mathbf{x}_0, 2R)$, the *physical scale* R is introduced via bounds on the distributional derivatives of f , where a test function ψ is a refined – smooth, non-negative, equal to 1 on $B(\mathbf{x}_0, R)$ and featuring optimal bounds on the derivatives over the outer R -layer – cut-off function on $B(\mathbf{x}_0, 2R)$. (Uniformity in all scales dictates linearity of the length of the outer layer in R ; hence $B(\mathbf{x}_0, R + R)$.)

More explicitly,

$$|(D^\alpha f, \psi)| \leq \int_{B(\mathbf{x}_0, 2R)} |f| |D^\alpha \psi| \leq \left(c(\alpha) \frac{1}{R^{|\alpha|}} |f|, \psi^{\rho(\alpha)} \right) \tag{1.1}$$

for some $c(\alpha) > 0$ and $\rho(\alpha)$ in $(0, 1)$.

(An attempt to introduce a concept of scale via characteristic functions in place of smooth cut-off functions would lead to infinite concentration – delta functions – invalidating much of the desired calculus.)

This approach has a similar flavor as introducing the *Fourier scale* $|\xi|$ via

$$\widehat{D^\alpha f}(\xi) = i^{|\alpha|} \xi^\alpha \hat{f}(\xi)$$

(in the Schwarz space, and then by duality in the space of tempered distributions).

Let \mathbf{x}_0 be in $B(\mathbf{0}, R_0)$ (R_0 being the integral scale, $B(\mathbf{0}, 2R_0) \subset \Omega$, where Ω is the global spatial domain) and $0 < R \leq R_0$. Define local – per unit of mass – kinetic energy, e and enstrophy, E , at time t , associated with the ball $B(\mathbf{x}_0, R)$ by

$$e_{\mathbf{x}_0, R}(t) = \int \frac{1}{2} |\mathbf{u}|^2 \phi^{2\delta-1} d\mathbf{x},$$

$$E_{\mathbf{x}_0, R}(t) = \int |\nabla \otimes \mathbf{u}|^2 \phi d\mathbf{x},$$

where $\phi = \eta \psi$ and η and ψ are refined cut-off functions in time and space, respectively (for some $\frac{1}{2} < \delta < 1$).

A total flux – (kinetic) energy plus pressure – through the boundary of a region D is given by

$$\int_{\partial D} \left(\frac{1}{2} |\mathbf{u}|^2 + p \right) \mathbf{u} \cdot \mathbf{n} ds = \int_D [(\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p] \cdot \mathbf{u} d\mathbf{x},$$

where \mathbf{n} is an outward normal. Considering the NSE localized to $B(\mathbf{x}_0, 2R)$ – and utilizing $\nabla \cdot \mathbf{u} = 0$ – leads to a localized flux,

$$\Phi_{\mathbf{x}_0, R}(t) = \int \left(\frac{1}{2} |\mathbf{u}|^2 + p \right) \mathbf{u} \cdot \nabla \phi d\mathbf{x} = - \int [(\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p] \cdot \mathbf{u} \phi d\mathbf{x}.$$

Since ψ can be constructed such that $\nabla \phi = \eta \nabla \psi$ is oriented along the radial directions of $B(\mathbf{x}_0, 2R)$ toward the center of the ball, $\Phi_{\mathbf{x}_0, R}$ represents the flux into $B(\mathbf{x}_0, R)$ through the layer between the spheres $S(\mathbf{x}_0, 2R)$ and $S(\mathbf{x}_0, R)$ ($\nabla \phi \equiv 0$ on $B(\mathbf{x}_0, R)$).

A more dynamic physical significance of the sign of $\Phi_{\mathbf{x}_0, R}$ can be seen from the equations: multiplying the NSE by $\psi \mathbf{u}$ and integrating over $B(\mathbf{x}_0, 2R)$ (formally, assuming smoothness) leads to

$$\frac{d}{dt} \int \frac{1}{2} |\mathbf{u}|^2 \psi d\mathbf{x} = \Phi_{\mathbf{x}_0, R} + \nu \int \Delta \mathbf{u} \cdot \mathbf{u} \psi d\mathbf{x}.$$

Plainly, the positivity of $\Phi_{\mathbf{x}_0, R}$ contributes to the increase of the kinetic energy around the point \mathbf{x}_0 at scale R .

Since the flux consists of both the kinetic and the pressure parts, a natural question is whether there is a transfer of the kinetic energy from larger scales into $B(\mathbf{x}_0, R)$, or perhaps the increase is mainly due to the change in pressure. In general, it is possible that the increase of the kinetic energy around \mathbf{x}_0 is due solely to the pressure part; a simple example being $\mathbf{u} = (c, c, c)t$, $p = cx_1 + cx_2 + cx_3$. However, *in physical situations where the kinetic energy on the (global) spatial domain Ω is non-increasing*, e.g., a bounded domain with no-slip boundary conditions, or the whole space with either decay at infinity or periodic boundary conditions (here, we are concerned with the case of decaying turbulence, setting the driving force to zero), *the increase of the kinetic energy in $B(\mathbf{x}_0, R)$ – and consequently, the positivity of $\Phi_{\mathbf{x}_0, R}$ – implies local transfer of the kinetic energy from larger scales* simply because the local kinetic energy is increasing while the global kinetic energy is non-increasing resulting in decrease of the kinetic energy in the complement. This is also consistent with the fact that in the aforementioned scenarios one can project the NSE – in an appropriate functional space – to the subspace of divergence-free functions effectively eliminating the pressure and revealing that the local flux $\Phi_{\mathbf{x}_0, R}$ is indeed driven by transport/inertial effects rather than the change in the pressure. (The $\mathbf{u} = (c, c, c)t$, $p = cx_1 + cx_2 + cx_3$ example pertains to a completely opposite situation; the kinetic energy is simply uniformly growing over the whole spatial domain.)

Henceforth, following the discussion in the preceding paragraphs – in the setting of decaying turbulence (zero driving force, non-increasing global energy) – the positivity and the negativity of $\Phi_{\mathbf{x}_0, R}$ will be interpreted as transfer of (kinetic) energy around the point \mathbf{x}_0 at scale R toward smaller scales and transfer of (kinetic) energy around the point \mathbf{x}_0 at scale R toward larger scales, respectively.

We also consider finite time averages for each of the aforementioned quantities.

Completely analogous definitions hold for shells of radii R and $2R$.

Our goal is to obtain a manifestation of the (kinetic) energy cascade in *physical space*, i.e., formulate a condition on $B(\mathbf{0}, R_0)$ that would imply that the time-averaged energy transfers/cascades to smaller scales across a range of scales (the existence of the inertial range).

A key point here is that we do not assume *any homogeneity* of the flow; hence one can not expect to show that the local fluxes are positive for each individual ball $B(\mathbf{x}, R)$. The best one can hope for is to prove the positivity of the flux over *some spatial average*.

We choose to work with a very straightforward spatial average: the arithmetic mean of the local fluxes – time-averaged, per unit mass – computed over a family of coverings of $B(\mathbf{0}, R_0)$, the so-called optimal coverings.

Let K_1 and K_2 be two positive integers. A covering $\{B(\mathbf{x}_i, R)\}_{i=1}^n$ of $B(\mathbf{0}, R_0)$ is an *optimal covering* (with parameters K_1 and K_2) if

$$\left(\frac{R_0}{R}\right)^3 \leq n \leq K_1 \left(\frac{R_0}{R}\right)^3,$$

and any point \mathbf{x} in $B(\mathbf{0}, R_0)$ is covered by at most K_2 balls $B(\mathbf{x}_i, 2R)$. (Optimal coverings exist for all large enough K_1 and K_2 , the critical values depending only on the dimension of the space. In \mathbb{R}^3 , we can take $K_1 = K_2 = 8$.)

Let f be a sign-varying quantity (e.g., the flux density $-(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p$) $\cdot \mathbf{u}$, and consider the arithmetic mean of the quantity locally averaged over the (optimal) covering

elements $B(\mathbf{x}_i, R)$,

$$F_R = \frac{1}{n} \sum_{i=1}^n \frac{1}{R^3} \int_{B(\mathbf{x}_i, 2R)} f \psi_i^\rho d\mathbf{x}$$

(for some $0 < \rho \leq 1$).

A revealing observation is that $F_R \sim \text{const}(R)$ for *all* optimal coverings at scale R (K_1 and K_2 fixed) indicates there are no significant fluctuations of the sign of f at scales comparable or greater than R . In other words, if there are significant fluctuations of the sign of f at scale R^* , F_R will run over a wide range of values while the average is being run over all permissible optimal coverings (determined by K_1 and K_2), for any R comparable or less than R^* .

When there is no change of the sign at all, i.e., in the case of a signed quantity (e.g., the energy density $f = \frac{1}{2}|\mathbf{u}|^2$, $\rho = 2\delta - 1$ or the enstrophy density $f = |\nabla \otimes \mathbf{u}|^2$, $\rho = 1$), one would then expect that for any scale R , $0 < R \leq R_0$, the averages F_R are all comparable to each other. This is in fact true (an easy proof).

Utilizing the NSE *via the local energy inequality* – in the mathematical setting of suitable weak solutions [1, 17] – we establish the positivity and near-constancy (comparable to νE , where ν is the viscosity and E is the average enstrophy over $B(\mathbf{0}, 2R_0) \times (0, 2T)$) of the averaged flux across a range of scales under a very simple and natural (in the sense of turbulence phenomenology) condition; namely, that Taylor micro-scale τ_0 associated with $B(\mathbf{0}, R_0)$ is smaller than the integral spatial scale R_0 (cf. (4.13)). The larger the gap, the deeper the inertial range. This condition is reminiscent of the Poincaré inequality on a domain of the corresponding size (see Remark 4.2); moreover, the condition in hand would be easy to check in physical experiments as the averages involved are very straightforward. In addition, the length of the time interval T is consistent with the intrinsic scaling of the model (cf. (4.2)).

It is interesting to interpret the cascade in the light of the above observation regarding the meaning of the near-constancy of optimal cover averages. Essentially, for any $\tau_0 \leq R \leq R_0$ (within the inertial range), the flux density does not experience significant fluctuations of sign at scale R ; the significant fluctuations of sign are only possible at the scales substantially smaller than τ_0 , i.e., inside the dissipation range.

The second part of the paper concerns *the locality* of the flux. It is believed (see [16]) that the energy flux inside the inertial range of turbulent flows depends strongly on the flow in nearby scales, its dependence on the lower and much higher scales being weak. The theoretical proof of this conjecture remained elusive. The first quantitative results on fluxes were obtained by the early 70's (see [13]). Much later, the authors in [15] used the NSE in the Fourier setting to explore locality of scale interactions for statistical averages, while the investigation in [5] revealed the locality of filtered energy flux under an assumption that solutions to the vanishing viscosity Euler's equations saturate a defining inequality of a suitable Besov space, i.e., under a (weak) scaling assumption. A more recent work [2] provided a proof of the locality of the energy flux in the setting of the Littlewood-Paley decomposition.

In the last section we prove the locality of the energy cascade – in decaying turbulence – in the physical space throughout the inertial range established in Theorem 4.1. In particular, considering dyadic shells at the scales $2^k R$ (k an integer) in the physical space, we show that both ultraviolet and infrared locality propagate *exponentially* in the shell number k .

To the best of our knowledge, the condition (4.13) is presently the only condition (in any solution setting) implying both the existence of the inertial range and the locality of

the energy flux. Moreover, it does not involve any additional regularity or homogeneity/scaling assumptions on the solutions to the NSE.

2. Preliminaries

We consider three dimensional incompressible Navier-Stokes equations (NSE)

$$\begin{aligned} \frac{\partial}{\partial t} \mathbf{u}(t, \mathbf{x}) - \nu \Delta \mathbf{u}(t, \mathbf{x}) + (\mathbf{u}(t, \mathbf{x}) \cdot \nabla) \mathbf{u}(t, \mathbf{x}) + \nabla p(t, \mathbf{x}) &= 0, \\ \nabla \cdot \mathbf{u}(t, \mathbf{x}) &= 0, \end{aligned} \tag{2.1}$$

where the space variable \mathbf{x} is in \mathbb{R}^3 and the time variable t is in $(0, \infty)$. The vector-valued function \mathbf{u} and the scalar-valued function p represent the fluid velocity and the pressure, respectively, while the constant ν is the viscosity of the fluid.

Since our goal is to investigate local fluxes in the physical space, the class of suitable weak solutions (see [1, 14]) will provide an appropriate mathematical framework.

Definition 2.1. *Let Ω be an open connected set in \mathbb{R}^3 . We say that (\mathbf{u}, p) is a suitable weak solution on $(0, \infty) \times \Omega$ if*

- (a) $\mathbf{u} \in L^\infty((0, \infty), L^2(\Omega)^3) \cap L^2((0, \infty), H^1(\Omega)^3)$ and $p \in L^{3/2}((0, \infty) \times \Omega)$;
- (b) the NSE (2.1) are satisfied in the weak (distributional) sense;
- (c) the local energy inequality is satisfied: for any $\phi \in \mathcal{D}((0, \infty) \times \Omega)$, $\phi \geq 0$ we have

$$\begin{aligned} 2\nu \iint |\nabla \otimes \mathbf{u}|^2 \phi \, d\mathbf{x} \, dt &\leq \iint |\mathbf{u}|^2 (\partial_t \phi + \nu \Delta \phi) \, d\mathbf{x} \, dt \\ &+ \iint (|\mathbf{u}|^2 + 2p) \mathbf{u} \cdot \nabla \phi \, d\mathbf{x} \, dt, \end{aligned} \tag{2.2}$$

where $\mathcal{D}((0, \infty) \times \Omega)$ denotes the space of infinitely differentiable functions with compact support in $(0, \infty) \times \Omega$.

The existence of the suitable weak solutions in the case where $\Omega = \mathbb{R}^3$ and the external force is zero, given a divergence-free initial condition in L^2 , was first established in [17]. See also [1, 14] for more general results related to existence and regularity properties of the suitable weak solutions.

A solution to the NSE on $(0, \infty) \times \Omega$ is called *regular* if its H^1 norm is bounded on $(0, T)$ for any T positive. Given appropriate boundary conditions, this implies that the solution is infinitely differentiable (in fact, analytic) in both space and time and so it is a classical physical solution. In particular, the local energy *equality* holds ((2.2) becomes an equality). The smoothness of the suitable weak solutions to the NSE is still an open problem, and the best result in this direction reads that the one-dimensional (parabolic) Hausdorff measure of the singular set in $(0, T) \times \Omega$ is zero [1] (outside the singular set, a suitable weak solution is infinitely differentiable in the spatial variables).

In what follows, we consider

$$R_0 > 0 \text{ such that } B(\mathbf{0}, 3R_0) \subset \Omega, \tag{2.3}$$

where $B(\mathbf{0}, 3R_0)$ denotes the ball in \mathbb{R}^3 centered at the origin and with the radius $3R_0$.

Let $1/2 \leq \delta < 1$. Choose $\psi_0 \in \mathcal{D}(B(\mathbf{0}, 2R_0))$ satisfying

$$0 \leq \psi_0 \leq 1, \quad \psi_0 = 1 \text{ on } B(\mathbf{0}, R_0), \quad \frac{|\nabla \psi_0|}{\psi_0^\delta} \leq \frac{C_0}{R_0}, \quad \frac{|\Delta \psi_0|}{\psi_0^{2\delta-1}} \leq \frac{C_0}{R_0^2}. \tag{2.4}$$

For a $T > 0$ (to be chosen later), $\mathbf{x}_0 \in B(\mathbf{0}, R_0)$ and $0 < R \leq R_0$ define $\phi = \phi_{\mathbf{x}_0, T, R}(t, \mathbf{x}) = \eta(t)\psi(\mathbf{x})$ to be used in the local energy inequality (2.2) where $\eta = \eta_T(t)$ and $\psi = \psi_{\mathbf{x}_0, R}(\mathbf{x})$ are refined cut-off functions satisfying the following conditions,

$$\eta \in \mathcal{D}(0, 2T), \quad 0 \leq \eta \leq 1, \quad \eta = 1 \text{ on } (T/4, 5T/4), \quad \frac{|\eta'|}{\eta^\delta} \leq \frac{C_0}{T}; \quad (2.5)$$

if $B(\mathbf{x}_0, R) \subset B(\mathbf{0}, R_0)$, then $\psi \in \mathcal{D}(B(\mathbf{x}_0, 2R))$ with

$$0 \leq \psi \leq \psi_0, \quad \psi = 1 \text{ on } B(\mathbf{x}_0, R) \cap B(\mathbf{0}, R_0), \quad \frac{|\nabla \psi|}{\psi^\delta} \leq \frac{C_0}{R}, \quad \frac{|\Delta \psi|}{\psi^{2\delta-1}} \leq \frac{C_0}{R^2}, \quad (2.6)$$

and if $B(\mathbf{x}_0, R) \not\subset B(\mathbf{0}, R_0)$, then $\psi \in \mathcal{D}(B(\mathbf{0}, 2R_0))$ with $\psi = 1$ on $B(\mathbf{x}_0, R) \cap B(\mathbf{0}, R_0)$ satisfying, in addition to (2.6), the following:

$$\psi = \psi_0 \text{ on the part of the cone in } \mathbb{R}^3 \text{ centered at zero and passing through } S(\mathbf{0}, R_0) \cap B(\mathbf{x}_0, R) \text{ between } S(\mathbf{0}, R_0) \text{ and } S(\mathbf{0}, 2R_0) \quad (2.7)$$

and

$$\psi = 0 \text{ on } B(\mathbf{0}, R_0) \setminus B(\mathbf{x}_0, 2R) \text{ and outside the part of the cone in } \mathbb{R}^3 \text{ centered at zero and passing through } S(\mathbf{0}, R_0) \cap B(\mathbf{x}_0, 2R) \text{ between } S(\mathbf{0}, R_0) \text{ and } S(\mathbf{0}, 2R_0). \quad (2.8)$$

Figure 1 illustrates the definition of ψ in the case $B(\mathbf{x}_0, R)$ is not entirely contained in $B(\mathbf{0}, R_0)$.

Remark 2.1. The additional conditions on the boundary elements (2.7) and (2.8) are necessary to obtain the lower bound on the fluxes in terms of the same version of the localized enstrophy E in Theorems 4.1 and 5.2 (see Remarks 4.4 and 5.5).

3. Localized Energy, Enstrophy and Flux; Ensemble Averages

Let $\mathbf{x}_0 \in B(\mathbf{0}, R_0)$ and $0 < R \leq R_0$. Define localized energy, e , and enstrophy, E , at time t – all per unit of mass – associated with $B(\mathbf{x}_0, R)$ by

$$e_{\mathbf{x}_0, R}(t) = \int \frac{1}{2} |\mathbf{u}|^2 \phi^{2\delta-1} d\mathbf{x}, \quad (3.1)$$

$$E_{\mathbf{x}_0, R}(t) = \int |\nabla \otimes \mathbf{u}|^2 \phi d\mathbf{x} \quad (3.2)$$

(for some $\frac{1}{2} < \delta < 1$).

The total – (kinetic) energy plus pressure – flux through sphere $S(\mathbf{x}_0, R)$ is given by

$$\int_{S(\mathbf{x}_0, R)} \left(\frac{1}{2} |\mathbf{u}|^2 + p \right) \mathbf{u} \cdot \mathbf{n} ds = \int_{B(\mathbf{x}_0, R)} [(\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p] \cdot \mathbf{u} d\mathbf{x},$$

where \mathbf{n} is an outward normal. Considering the NSE localized to $B(\mathbf{x}_0, R)$ leads to a localized version of the flux,

$$\Phi_{\mathbf{x}_0, R}(t) = \int \left(\frac{1}{2} |\mathbf{u}|^2 + p \right) \mathbf{u} \cdot \nabla \phi d\mathbf{x} = - \int [(\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p] \cdot \mathbf{u} \phi d\mathbf{x}, \quad (3.3)$$

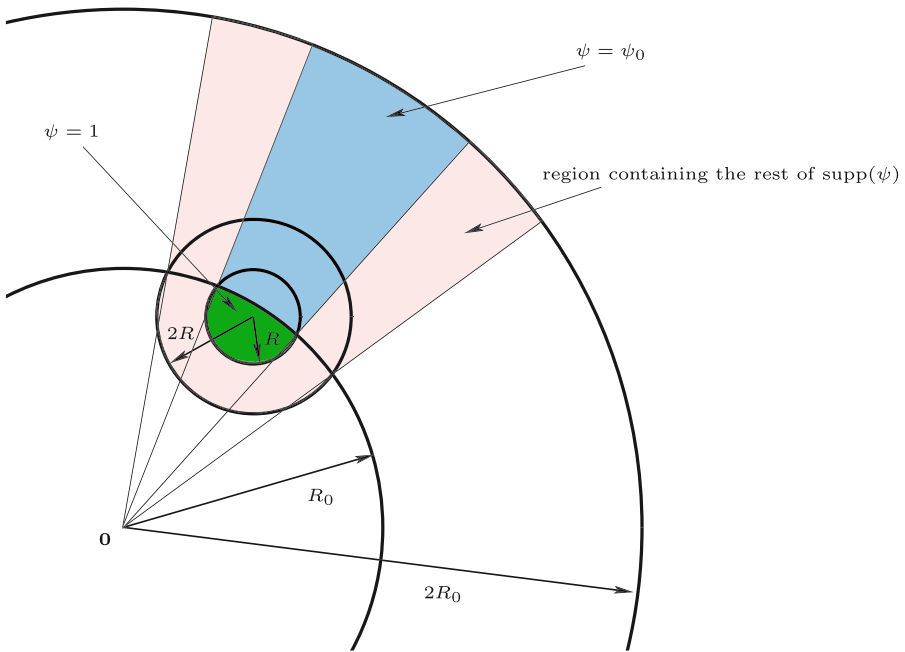


Fig. 1. Regions of $\text{supp}(\psi)$ in the case $B(\mathbf{x}_0, R) \not\subset B(\mathbf{0}, R_0)$, cross-section

where $\phi = \eta\psi$ with η and ψ as in (2.5–2.6). Since ψ can be constructed such that $\nabla\phi = \eta\nabla\psi$ is oriented along the radial directions of $B(\mathbf{x}_0, R)$ towards the center of the ball \mathbf{x}_0 , $\Phi(\mathbf{x}_0, R)$ represents the flux *into* $B(\mathbf{x}_0, R)$ through the layer between the spheres $S(\mathbf{x}_0, 2R)$ and $S(\mathbf{x}_0, R)$ (in the case of the boundary elements satisfying the additional hypotheses (2.7) and (2.8), ψ is almost radial and the gradient still points inward).

For a quantity $\Theta_{\mathbf{x},R}(t)$, $t \in [0, 2T]$ and a covering $\{B(\mathbf{x}_i, R)\}_{i=1,n}$ of $B(\mathbf{0}, R_0)$ define a *time-space ensemble average*

$$\langle \Theta \rangle_R = \frac{1}{T} \int \frac{1}{n} \sum_{i=1}^n \frac{1}{R^3} \Theta_{\mathbf{x}_i, R}(t) dt. \tag{3.4}$$

Denote by

$$e_R = \langle e_{\mathbf{x},R}(t) \rangle_R, \tag{3.5}$$

$$E_R = \langle E_{\mathbf{x},R}(t) \rangle_R, \tag{3.6}$$

$$\Phi_R = \langle \Phi_{\mathbf{x},R}(t) \rangle_R, \tag{3.7}$$

the averaged localized energy, enstrophy and inward-directed flux over balls of radius R covering $B(\mathbf{0}, R_0)$.

Also, introduce the time-space average of the localized energy on $B(\mathbf{0}, R_0)$,

$$e = \frac{1}{T} \int \frac{1}{R_0^3} e_{\mathbf{0},R_0}(t) dt = \frac{1}{T} \frac{1}{R_0^3} \iiint \frac{1}{2} |\mathbf{u}|^2 \phi_0^{2\delta-1} d\mathbf{x} dt \tag{3.8}$$

and the time-space average of the localized enstrophy on $B(\mathbf{0}, R_0)$,

$$E = \frac{1}{T} \int \frac{1}{R_0^3} E_{\mathbf{0}, R_0}(t) dt = \frac{1}{T} \frac{1}{R_0^3} \iint |\nabla \otimes \mathbf{u}|^2 \phi_0 d\mathbf{x} dt, \tag{3.9}$$

where

$$\phi_0(t, \mathbf{x}) = \eta(t)\psi_0(\mathbf{x}) \tag{3.10}$$

with ψ_0 defined in (2.4).

Finally, define Taylor length scale associated with $B(\mathbf{0}, R_0)$ by

$$\tau_0 = \left(\frac{e}{E}\right)^{1/2}. \tag{3.11}$$

Note that the possible lack of regularity may produce additional loss of energy, resulting in *anomalous* energy dissipation and the loss of flux leading to the strict inequality in (2.2). Let us mention here that in the turbulence literature the term ‘anomalous dissipation’ is usually utilized in the context of the possible energy dissipation due to the (possible) singularities in the 3D Euler equations (the observation originally made by Onsager); for rigorous results on Onsager’s conjecture on the energy conservation in the Euler equations see, e.g., [3], and a recent work [2].

Denote by $\Phi_{\mathbf{x}_0, R}^\infty$ the loss of flux due to possible singularities in $[0, 2T] \times B(\mathbf{x}_0, 2R)$,

$$\begin{aligned} & \iint \left(\frac{1}{2}|\mathbf{u}|^2 + p\right)\mathbf{u} \cdot \nabla \phi d\mathbf{x} dt - \Phi_{\mathbf{x}_0, R}^\infty \\ &= \nu \iint |\nabla \otimes \mathbf{u}|^2 \phi d\mathbf{x} dt - \frac{1}{2} \iint |\mathbf{u}|^2 (\partial_t \phi + \nu \Delta \phi) d\mathbf{x} dt \end{aligned} \tag{3.12}$$

where $\phi = \eta\psi$ with η and ψ as in (2.5) and (2.6–2.8). In particular, denote by $\Phi_\infty = \Phi_{\mathbf{0}, R_0}^\infty$ the loss of flux due to singularities in $[0, 2T] \times B(\mathbf{0}, 2R_0)$.

We will also consider the time-space ensemble averages of these *anomalous* fluxes,

$$\Phi_R^\infty = \frac{1}{n} \sum_{i=1}^n \frac{1}{T} \frac{1}{R^3} \Phi_{\mathbf{x}_i, R}^\infty. \tag{3.13}$$

Note that due to (2.2), all the anomalous fluxes are nonnegative,

$$\Phi_{\mathbf{x}_0, R}^\infty \geq 0, \quad \Phi_\infty \geq 0, \quad \Phi_R^\infty \geq 0; \tag{3.14}$$

they are all zero provided the equality holds in (2.2) inside $[0, 2T] \times B(\mathbf{0}, 2R_0)$. In particular, the anomalous fluxes are all zero provided the solution in view is *regular* on $[0, 2T] \times B(\mathbf{0}, 2R_0)$.

Consequently, the total localized flux into $B(\mathbf{x}_0, R)$ over interval $[0, 2T]$, including the (loss of) flux due to the possible loss of regularity, is

$$\Psi_{\mathbf{x}_0, R} = \int \Phi_{\mathbf{x}_0, R}(t) dt - \Phi_{\mathbf{x}_0, R}^\infty, \tag{3.15}$$

and the time-space ensemble average of this flux at scale R is

$$\Psi_R = \Phi_R - \Phi_R^\infty. \tag{3.16}$$

We will refer to $\Psi_{\mathbf{x}_0, R}$ and Ψ_R as the *modified* flux over $[0, 2T]$ into $B(\mathbf{x}_0, R)$ and the (time-space ensemble) averaged *modified* flux at the scale R , respectively.

Let $K_1, K_2 > 1$ be two positive integers (independent of R, R_0 , and any of the parameters of the NSE).

Definition 3.1. We say that a covering of $B(\mathbf{0}, R_0)$ by n balls of radius R is optimal if

$$\left(\frac{R_0}{R}\right)^3 \leq n \leq K_1 \left(\frac{R_0}{R}\right)^3; \tag{3.17}$$

$$\text{any } \mathbf{x} \in B(\mathbf{0}, R_0) \text{ is covered by at most } K_2 \text{ balls } B(\mathbf{x}_i, 2R). \tag{3.18}$$

Note that optimal coverings exist for any $0 < R \leq R_0$ provided K_1 and K_2 are large enough. In fact, the choice of K_1 and K_2 depends only on dimension of the space; in \mathbb{R}^3 we can choose $K_1 = K_2 = 8$.

Henceforth, we assume that the averages $\langle \cdot \rangle_R$ are taken with respect to optimal coverings.

Lemma 3.1. If the covering $\{B(\mathbf{x}_i, R)\}_{i=1,n}$ of $B(\mathbf{0}, R_0)$ is optimal then

$$\Phi_R^\infty \leq K \frac{1}{T} \frac{1}{R_0^3} \Phi_\infty, \tag{3.19}$$

where $K > 0$ is a constant depending only on K_2 and dimension of the space \mathbb{R}^3 .

Proof. Let $\{\mathbf{x}_j\}$ be a subset of $\{\mathbf{x}_i\}_{i=1,n}$ such that interiors of the balls $B(\mathbf{x}_j, 2R)$ are pairwise disjoint. Using (3.12), we obtain

$$\begin{aligned} & \iint \left(\frac{1}{2}|\mathbf{u}|^2 + p\right)\mathbf{u} \cdot \nabla \phi_0 \, d\mathbf{x} \, dt - \Phi_\infty \\ &= \nu \iint |\nabla \otimes \mathbf{u}|^2 \phi_0 \, d\mathbf{x} \, dt - \frac{1}{2} \iint |\mathbf{u}|^2 (\partial_t \phi_0 + \nu \Delta \phi_0) \, d\mathbf{x} \, dt \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} & \iint \left(\frac{1}{2}|\mathbf{u}|^2 + p\right)\mathbf{u} \cdot \nabla \left(\sum_j \phi_{i_j}\right) \, d\mathbf{x} \, dt - \sum_j \Phi_{\mathbf{x}_{i_j}, R}^\infty \\ &= \nu \iint |\nabla \otimes \mathbf{u}|^2 \left(\sum_j \phi_{i_j}\right) \, d\mathbf{x} \, dt - \frac{1}{2} \iint |\mathbf{u}|^2 [\partial_t \left(\sum_j \phi_{i_j}\right) + \nu \Delta \left(\sum_j \phi_{i_j}\right)] \, d\mathbf{x} \, dt, \end{aligned} \tag{3.21}$$

where $\phi_0 = \eta \psi_0$ and $\phi_{i_j} = \eta \psi_{i_j}$ with η as in (2.5), ψ_0 as in (2.4) and ψ_{i_j} a test function corresponding to $B(\mathbf{x}_{i_j}, R)$ satisfying (2.6–2.8).

Note that the definitions of ϕ_0 and ϕ_{i_j} imply

$$\tilde{\phi} = \phi_0 - \sum_j \phi_{i_j} \geq 0;$$

hence, by the local energy inequality (2.2),

$$\begin{aligned} & \iint \left(\frac{1}{2}|\mathbf{u}|^2 + p\right)\mathbf{u} \cdot \nabla \tilde{\phi} \, d\mathbf{x} \, dt \\ & \geq \nu \iint |\nabla \otimes \mathbf{u}|^2 \tilde{\phi} \, d\mathbf{x} \, dt - \frac{1}{2} \iint |\mathbf{u}|^2 (\partial_t \tilde{\phi} + \nu \Delta \tilde{\phi}) \, d\mathbf{x} \, dt. \end{aligned} \tag{3.22}$$

If we add relations (3.21) and (3.22) and then subtract (3.20) we obtain

$$\Phi_\infty \geq \sum_j \Phi_{\mathbf{x}_{i_j}, R}^\infty. \tag{3.23}$$

Let \mathcal{L} be a cubic lattice inside $B(\mathbf{0}, R_0)$ with the points situated at the vertices of cubes of side $R/2$ (Note that this lattice can be chosen such that the number of points in it is between $2^3(R_0/R)^3$ and $(4\pi/3)2^3(R_0/R)^3$.)

Since the covering $\{B(\mathbf{x}_i, R)\}$ is optimal, each point in \mathcal{L} is contained in at most K_2 balls. Moreover, any ball in the covering will contain at least one point from the lattice.

If \mathcal{L}' is a sub-lattice of \mathcal{L} with points at vertices of cubes of side $4R$, then the interiors of balls of radius $2R$ containing different points of \mathcal{L}' are pairwise disjoint, and thus if we denote by $B(\mathbf{x}_p, R)$ a ball from the covering $\{B(\mathbf{x}_i, R)\}$ containing the point $p \in \mathcal{L}'$, by (3.22),

$$\Phi_\infty \geq \sum_{p \in \mathcal{L}'} \Phi_{\mathbf{x}_{i_p}, R}^\infty.$$

Note that for each point $p \in \mathcal{L}'$ there are at most K_2 choices for $B(\mathbf{x}_{i_p}, R)$. So

$$K_2 \Phi_\infty \geq \sum_{i: B(\mathbf{x}_i, R) \cap \mathcal{L}' \neq \emptyset} \Phi_{\mathbf{x}_i, R}^\infty.$$

Clearly \mathcal{L} can be written as a union of $8^3 = 256$ sub-lattices $\mathcal{L}'_k, k = 1, \dots, 256$, each \mathcal{L}'_k having the same properties as \mathcal{L}' . Thus,

$$8^3 K_2 \Phi_\infty \geq \sum_{i=1}^n \Phi_{\mathbf{x}_i, R}^\infty.$$

Consequently,

$$\Phi_R^\infty = \frac{1}{T} \frac{1}{R^3} \frac{1}{n} \sum_{i=1}^n \Phi_{\mathbf{x}_i, R}^\infty \leq 8^3 K_2 \frac{1}{T} \frac{1}{R^3} \frac{1}{n} \Phi_\infty \leq 8^3 K_2 \frac{1}{T} \frac{1}{R_0^3} \Phi_\infty,$$

where the last inequality is due to n satisfying (3.18). \square

According to the lemma, the time-space ensemble averages Φ_R^∞ taken over the optimal coverings at the scale R are bounded, *independently* of R , by the average loss of flux due to possible singularities inside $B(\mathbf{0}, 2R_0)$.

4. Energy Cascade

Let $\{B(\mathbf{x}_i, R)\}_{i=1, n}$ be an optimal covering of $B(\mathbf{0}, R_0)$.

Note that the local energy equality (3.12) and the definitions of E_R, Φ_R and Φ_R^∞ ((3.6), (3.7) and (3.13)) imply

$$\Psi_R = \Phi_R - \Phi_R^\infty = \nu E_R - \frac{1}{n} \sum_{i=1}^n \frac{1}{T} \frac{1}{R^3} \iint \frac{1}{2} |\mathbf{u}|^2 (\partial_t \phi_i + \nu \Delta \phi_i) d\mathbf{x} dt, \tag{4.1}$$

where $\phi_i = \eta\psi_i$ and ψ_i is the spatial cut-off on $B(x_i, 2R)$ satisfying (2.5–2.8).

If

$$T \geq \frac{R_0^2}{\nu}, \tag{4.2}$$

then for any $0 < R \leq R_0$,

$$\begin{aligned} |(\phi_i)_t| &= |\eta_t \psi_i| \leq C_0 \frac{1}{T} \eta^\delta \psi_i \leq \nu \frac{C_0}{R^2} \phi_i^{2\delta-1}, \\ \nu |\Delta \phi_i| &= \nu |\eta \Delta \psi_i| \leq C_0 \frac{\nu}{R^2} \eta \psi_i^{2\delta-1} \leq \nu \frac{C_0}{R^2} \phi_i^{2\delta-1}; \end{aligned}$$

hence,

$$\Psi_R \geq \nu E_R - \nu \frac{C_0}{R^2} e_R.$$

The optimality conditions (3.17) and (3.18) paired with (2.6–2.8) imply

$$E_R \geq \frac{1}{K_1} E \tag{4.3}$$

and

$$e_R \leq K_2 e. \tag{4.4}$$

Consequently,

$$\Psi_R \geq \nu \frac{1}{K_1} E - \nu \frac{C_0 K_2}{R^2} e \tag{4.5}$$

leading to the following proposition.

Proposition 4.1.

$$\Psi_R \geq c_1 \nu E \left(1 - c_2 \frac{\tau_0^2}{R^2} \right) \tag{4.6}$$

with $c_1 = 1/K_1$ and $c_2 = C_0 K_1 K_2$ (provided conditions (3.17–3.18) are satisfied).

Suppose that

$$\tau_0 < \frac{\gamma}{c_2^{1/2}} R_0 \tag{4.7}$$

for some $0 < \gamma < 1$. Then, for any $R, (c_2^{1/2}/\gamma) \tau_0 \leq R \leq R_0$,

$$\Psi_R \geq c_1 (1 - \gamma^2) \nu E = c_{0,\gamma} \nu E, \tag{4.8}$$

where

$$c_{0,\gamma} = c_1 (1 - \gamma^2) = \frac{1 - \gamma^2}{K_1}. \tag{4.9}$$

To obtain an upper bound on the averaged modified flux, note that for optimal coverings, in addition to (4.4),

$$E_R \leq K_2 E. \quad (4.10)$$

Hence, (4.1) implies

$$\Psi_R \leq \nu E_R + \nu \frac{C_0}{R^2} e_R \leq \nu K_2 E + \nu C_0 K_2 \frac{1}{R^2} e.$$

If the condition (4.7) holds for some $0 < \gamma < 1$, then it follows that for any R , $(c_2^{1/2}/\gamma) \tau_0 \leq R \leq R_0$,

$$\Psi_R \leq \nu K_2 E + \nu \frac{C_0 K_2 \gamma^2}{c_2} E \leq c_{1,\gamma} \nu E, \quad (4.11)$$

where

$$c_{1,\gamma} = K_2 \left[1 + \frac{C_0 \gamma^2}{c_2} \right] = K_2 \left[1 + \frac{\gamma^2}{K_1 K_2} \right]. \quad (4.12)$$

Thus we have proved the following.

Theorem 4.1. *Assume that for some $0 < \gamma < 1$,*

$$\tau_0 < c\gamma R_0, \quad (4.13)$$

where

$$c = \frac{1}{\sqrt{C_0 K_1 K_2}}. \quad (4.14)$$

Then, for all R ,

$$\frac{1}{c\gamma} \tau_0 \leq R \leq R_0, \quad (4.15)$$

the averaged modified flux Ψ_R satisfies

$$c_{0,\gamma} \nu E \leq \Psi_R \leq c_{1,\gamma} \nu E, \quad (4.16)$$

where

$$c_{0,\gamma} = \frac{1 - \gamma^2}{K_1}, \quad c_{1,\gamma} = K_2 \left[1 + \frac{\gamma^2}{K_1 K_2} \right], \quad (4.17)$$

and the average $\langle \cdot \rangle_R$ is computed over a time interval $[0, 2T]$ with $T \geq R_0^2/\nu$ and determined by an optimal covering of $B(\mathbf{0}, R_0)$ (i.e., a covering satisfying (3.17) and (3.18)).

Remark 4.1. As noted in the Introduction – in the case the global energy is non-increasing – the theorem provides a sufficient condition for the energy cascade; i.e., a nearly constant nonlinear transfer of time-averaged (kinetic) energy to smaller scales across the inertial range defined by (4.15).

More precisely, since we are working with weak solutions, the expression for the rate of change of local kinetic energy (1.2) morphs into

$$-\iint \frac{1}{2} |\mathbf{u}|^2 \phi_t = \Psi_{\mathbf{x}_0, R} + \text{viscous terms.} \tag{4.18}$$

The interpretation remains the same, the only differences being that the time-derivative of the local kinetic energy is taken in the sense of distributions and the flux got replaced with the modified flux to account for possible singularities.

Note that the averages are taken over finite-time intervals $[0, 2T]$ with $T \geq R_0^2/\nu$ (see (4.2)). This lower bound on the length of the time interval T is consistent with the picture of decaying turbulence; namely, small ν corresponds to the well-developed turbulence which then persists for a longer time and it makes sense to average over longer time-intervals.

Remark 4.2. In the language of turbulence, the condition (4.13) simply reads that the Taylor *micro-scale* computed over the domain in view is smaller than the *integral scale* (diameter of the domain).

On the other hand, (4.13) is equivalent to

$$\frac{1}{T} \iint |\mathbf{u}|^2 \phi_0^{2\delta-1} d\mathbf{x} dt < \frac{\gamma^2}{C_0 K_1 K_2} R_0^2 \frac{1}{T} \iint |\nabla \otimes \mathbf{u}|^2 \phi_0 d\mathbf{x} dt$$

which can be read as a requirement that the time average of a Poincaré-like inequality on $B(\mathbf{0}, 2R_0)$ is not saturating; this will hold for a variety of flows in the regions of active fluid (large gradients).

Remark 4.3. Since $\Phi_R^\infty \geq 0$, all the *lower* bounds on the averaged modified fluxes hold for the usual averaged fluxes Φ_R ; in particular,

$$\Phi_R \geq c_1 \nu E \left(1 - c_2 \frac{\tau_0^2}{R^2} \right) \tag{4.19}$$

and consequently, provided (4.13) and (4.15) hold,

$$\Phi_R \geq c_{0,\gamma} \nu E. \tag{4.20}$$

Also, if a solution \mathbf{u} is such that the (2.2) holds with equality and in particular, if the solution is *regular*, then *all* the estimates, including (4.16), hold for the usual averaged flux Φ_R .

Remark 4.4. If we do not impose the additional assumptions (2.7) and (2.8) for the test functions on the balls $B(\mathbf{x}_i, R) \not\subset B(\mathbf{0}, R_0)$, then the lower bounds for Ψ_R in (4.6) and (4.16) will hold with E replaced by the time-space average of the *non-localized* in space enstrophy on $B(\mathbf{0}, R_0)$,

$$E' = \frac{1}{T} \int_0^{2T} \frac{1}{R_0^3} \int_{B(\mathbf{0}, R_0)} |\nabla \otimes \mathbf{u}|^2 \eta d\mathbf{x} dt.$$

This is the case because the estimate (4.3) gets replaced with

$$E_R \geq \frac{1}{K_1} E'.$$

5. Locality of the Averaged Flux

Let $\mathbf{x}_0 \in B(\mathbf{0}, R_0)$, $0 < R_2 < R_1 \leq R_0$. In order to study the flux through the shell $A(\mathbf{x}_0, R_1, R_2)$ between the spheres $S(\mathbf{x}_0, R_2)$ and $S(\mathbf{x}_0, R_1)$, in what follows, we will consider the modified test functions $\phi = \phi_{\mathbf{x}_0, T, R_1, R_2}(t, \mathbf{x}) = \eta(t)\psi(\mathbf{x})$ to be used in the local energy inequality (2.2), where $\eta = \eta_T(t)$ as in (2.5) and $\psi \in \mathcal{D}(A(\mathbf{x}_0, 2R_1, R_2/2))$ satisfying

$$0 \leq \psi \leq \psi_0, \quad \psi = 1 \quad \text{on } A(\mathbf{x}_0, R_1, R_2), \quad \frac{|\nabla\psi|}{\psi^\delta} \leq \frac{C_0}{\tilde{R}}, \quad \frac{|\Delta\psi|}{\psi^{2\delta-1}} \leq \frac{C_0}{\tilde{R}^2}, \quad (5.1)$$

where ψ_0 is defined in (2.4) and

$$\tilde{R} = \tilde{R}(R_1, R_2) = \min\{R_2, R_1 - R_2\}. \quad (5.2)$$

Use the ϕ above to define the localized time-averaged flux through the shell between the spheres $S(\mathbf{x}_0, R_2)$ and $S(\mathbf{x}_0, R_1)$ as

$$\Phi_{\mathbf{x}_0, R_1, R_2} = \frac{1}{T} \iint \left(\frac{1}{2} |\mathbf{u}|^2 + p \right) \mathbf{u} \cdot \nabla \phi \, d\mathbf{x} \, dt. \quad (5.3)$$

If $\Phi_{\mathbf{x}_0, R_1, R_2}^\infty$ is the anomalous flux inside $A(\mathbf{x}_0, 2R_1, R_2/2)$, i.e., if $\Phi_{\mathbf{x}_0, R_1, R_2}$ satisfies

$$\begin{aligned} & \iint \left(\frac{1}{2} |\mathbf{u}|^2 + p \right) \mathbf{u} \cdot \nabla \phi \, d\mathbf{x} \, dt - \Phi_{\mathbf{x}_0, R_1, R_2}^\infty \\ &= \nu \iint |\nabla \otimes \mathbf{u}|^2 \phi \, d\mathbf{x} \, dt - \frac{1}{2} \iint |\mathbf{u}|^2 (\partial_t \phi + \nu \Delta \phi) \, d\mathbf{x} \, dt, \end{aligned} \quad (5.4)$$

then define the time average of the modified localized fluxes through the shells $A(\mathbf{x}_0, R_1, R_2)$ by

$$\Psi_{\mathbf{x}_0, R_1, R_2} = \Phi_{\mathbf{x}_0, R_1, R_2} - \frac{1}{T} \Phi_{\mathbf{x}_0, R_1, R_2}^\infty. \quad (5.5)$$

As already mentioned, the modified fluxes can be viewed as total fluxes including the (loss of) flux due to possible singularities inside the shell. Also note that the local energy inequality (2.2) implies

$$\Phi_{\mathbf{x}_0, R_1, R_2}^\infty \geq 0. \quad (5.6)$$

Define the time-averaged energy and enstrophy in the shell between the spheres $S(\mathbf{x}_0, R_2)$ and $S(\mathbf{x}_0, R_1)$ by

$$\begin{aligned} e_{\mathbf{x}_0, R_1, R_2} &= \frac{1}{T} \iint \frac{1}{2} |\mathbf{u}|^2 \phi^{2\delta-1} \, d\mathbf{x} \, dt, \\ E_{\mathbf{x}_0, R_1, R_2} &= \frac{1}{T} \iint |\nabla \otimes \mathbf{u}|^2 \phi \, d\mathbf{x} \, dt; \end{aligned} \quad (5.7)$$

then,

$$\tau_{\mathbf{x}_0, R_1, R_2} = \left(\frac{e_{\mathbf{x}_0, R_1, R_2}}{E_{\mathbf{x}_0, R_1, R_2}} \right)^{1/2} \tag{5.8}$$

is the *local* Taylor length scale associated with the shell $A(\mathbf{x}_0, R_1, R_2)$.

Note that

$$\nu |\Delta \phi| = \nu |\eta \Delta \psi| \leq \nu \frac{C_0}{\tilde{R}^2} |\eta \phi^{2\delta-1}| \leq \nu \frac{C_0}{\tilde{R}^2} \phi^{2\delta-1} \tag{5.9}$$

and

$$|\phi_t| = |\eta_t \psi| \leq C_0 \frac{1}{T} \eta^\delta \psi \leq \nu \frac{C_0}{\tilde{R}^2} \phi^{2\delta-1}, \tag{5.10}$$

provided

$$T \geq \frac{R_0^2}{\nu} \left(\geq \frac{\tilde{R}^2}{\nu} \right). \tag{5.11}$$

Hence, (5.4) implies that for any $\mathbf{x}_0 \in B(\mathbf{0}, R_0)$ and any $0 < R_2 < R_1 \leq R_0$,

$$\begin{aligned} \Psi_{\mathbf{x}_0, R_1, R_2} &\geq \nu E_{\mathbf{x}_0, R_1, R_2} - \nu \frac{C_0}{\tilde{R}^2} e_{\mathbf{x}_0, R_1, R_2} \\ &= \nu E_{\mathbf{x}_0, R_1, R_2} \left(1 - C_0 \frac{\tau_{\mathbf{x}_0, R_1, R_2}^2}{\tilde{R}^2} \right) \end{aligned} \tag{5.12}$$

leading to the following proposition.

Proposition 5.1. *Let $0 < \gamma < 1$. Then, for any shell $A(\mathbf{x}_0, R_1, R_2)$ satisfying*

$$\tau_{\mathbf{x}_0, R_1, R_2} < \frac{\gamma}{C_0^{1/2}} \tilde{R} \tag{5.13}$$

with \tilde{R} defined by (5.2),

$$\Psi_{\mathbf{x}_0, R_1, R_2} \geq \nu E_{\mathbf{x}_0, R_1, R_2} (1 - \gamma^2). \tag{5.14}$$

Similarly, utilizing (5.4) again, we obtain an upper bound

$$\Psi_{\mathbf{x}_0, R_1, R_2} \leq \nu E_{\mathbf{x}_0, R_1, R_2} + \nu \frac{C_0}{\tilde{R}^2} e_{\mathbf{x}_0, R_1, R_2}.$$

If the condition (5.13) holds for some $0 < \gamma < 1$, then it follows that

$$\Psi_{\mathbf{x}_0, R_1, R_2} \leq \nu E_{\mathbf{x}_0, R_1, R_2} + \gamma^2 \nu E_{\mathbf{x}_0, R_1, R_2} \leq (1 + \gamma^2) \nu E_{\mathbf{x}_0, R_1, R_2}; \tag{5.15}$$

thus, we have arrived at our first locality result.

Theorem 5.1. *Let $0 < \gamma < 1$, $\mathbf{x}_0 \in B(\mathbf{0}, R_0)$ and $0 < R_2 < R_1 \leq R_0$. If*

$$\tau_{\mathbf{x}_0, R_1, R_2} < \frac{\gamma}{C_0^{1/2}} \tilde{R} \tag{5.16}$$

with \tilde{R} defined by (5.2), then

$$(1 - \gamma^2) \nu E_{\mathbf{x}_0, R_1, R_2} \leq \Psi_{\mathbf{x}_0, R_1, R_2} \leq (1 + \gamma^2) \nu E_{\mathbf{x}_0, R_1, R_2}, \tag{5.17}$$

where the time average is taken over an interval of time $[0, 2T]$ with $T \geq R_0^2/\nu$.

Remark 5.1. The theorem states that if the local Taylor scale $\tau_{\mathbf{x}_0, R_1, R_2}$, associated with a shell $A(\mathbf{x}_0, R_1, R_2)$, is smaller than the thickness of the shell \tilde{R} (a local integral scale), then the time average of the modified flux through that shell towards its center \mathbf{x}_0 is comparable to the time average of the localized enstrophy in the shell, $E_{\mathbf{x}_0, R_1, R_2}$. Thus, under the assumption (5.16) the flux through the shell $A(\mathbf{x}_0, R_1, R_2)$ depends essentially only on the enstrophy contained in the neighborhood of the shell, regardless of what happens at the other scales, making (5.13) a sufficient condition for the *locality* of the flux through $A(\mathbf{x}_0, R_1, R_2)$.

Remark 5.2. Similarly as in the case of condition (4.13), we can observe that condition (5.16) can be viewed as a requirement that the time average of a Poincaré-like inequality on the shell is not saturating, making it plausible in the case of intense fluid activity in a neighborhood of the shell.

Remark 5.3. Since $\Phi_{\mathbf{x}_0, R_1, R_2}^\infty \geq 0$, all the *lower* bounds on the modified fluxes hold for the usual fluxes $\Phi_{\mathbf{x}_0, R_0, R_1}$; in particular, we have

$$\Phi_{\mathbf{x}_0, R_1, R_2} \geq \nu E_{\mathbf{x}_0, R_1, R_2} \left(1 - C_0 \frac{\tau_{\mathbf{x}_0, R_1, R_2}^2}{\tilde{R}^2} \right) \tag{5.18}$$

and, provided (5.16) holds,

$$\Phi_{\mathbf{x}_0, R_0, R_1} \geq (1 - \gamma^2) \nu E_{\mathbf{x}_0, R_1, R_2}. \tag{5.19}$$

Also, if a solution \mathbf{u} is such that (2.2) holds with equality and in particular, if the solution is *regular*, then *all* the estimates, including (5.17), hold for the usual averaged flux $\Phi_{\mathbf{x}_0, R_1, R_2}$.

In order to further study the locality of the flux, we will estimate the ensemble averages of the fluxes through the shells $A(\mathbf{x}_i, 2R, R)$ of thickness $\tilde{R} = R$. Since we are interested in the shells inside $B(\mathbf{0}, R_0)$, we require the lattice points \mathbf{x}_i to satisfy

$$B(\mathbf{x}_i, R) \subset B(\mathbf{0}, R_0). \tag{5.20}$$

To each $A(\mathbf{x}_i, 2R, R)$ we associate a test function $\phi_i = \eta \psi_i$, where η satisfies (2.5) and ψ_i satisfies the following.

If $A(\mathbf{x}_i, 2R, R) \subset B(\mathbf{0}, R_0)$, then $\psi_i \in \mathcal{D}(A(\mathbf{x}_i, 4R, R/2))$ with

$$0 \leq \psi_i \leq \psi_0, \quad \psi_i = 1 \text{ on } A(\mathbf{x}_i, 2R, R) \cap B(\mathbf{0}, R_0), \quad \frac{|\nabla \psi_i|}{\psi_i^\delta} \leq \frac{C_0}{R}, \quad \frac{|\Delta \psi_i|}{\psi_i^{2\delta-1}} \leq \frac{C_0}{R^2}, \tag{5.21}$$

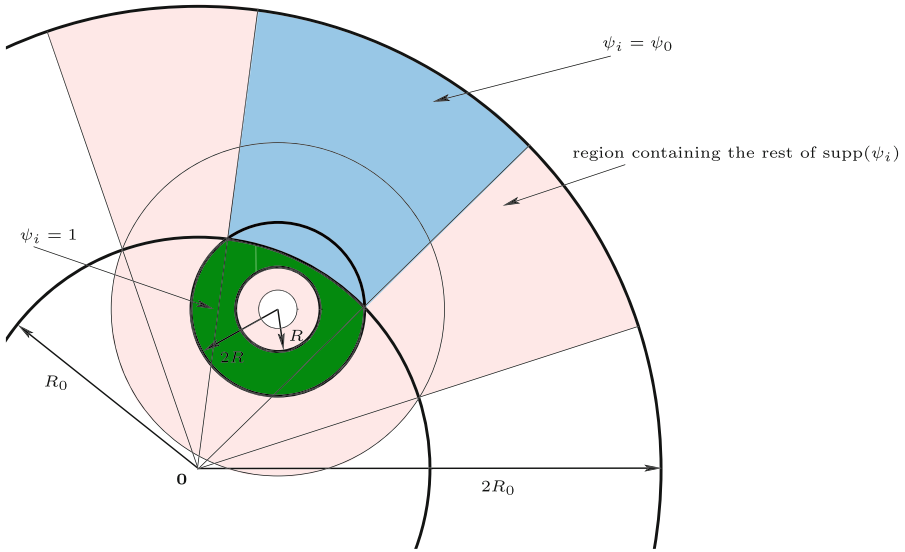


Fig. 2. A cross-section of regions of $\text{supp}(\psi_i)$ in the case $A(\mathbf{x}_i, 2R, R) \not\subset B(\mathbf{0}, R_0)$

and if $A(\mathbf{x}_i, 2R, R) \not\subset B(\mathbf{0}, R_0)$ (i.e. we have $B(\mathbf{x}_i, R) \subset B(\mathbf{0}, R_0)$ and $B(\mathbf{x}_i, 2R) \setminus B(\mathbf{0}, R_0) \neq \emptyset$), then $\psi_i \in \mathcal{D}(B(\mathbf{0}, 2R_0))$ with $\psi_i = 1$ on $A(\mathbf{x}_i, 2R, R) \cap B(\mathbf{0}, R_0)$ satisfying, in addition to (5.21), the following:

$$\psi_i = \psi_0 \text{ on the part of the cone in } \mathbb{R}^3 \text{ centered at zero and passing through } S(\mathbf{0}, R_0) \cap B(\mathbf{x}_i, 2R) \text{ between } S(\mathbf{0}, R_0) \text{ and } S(\mathbf{0}, 2R_0) \quad (5.22)$$

and

$$\psi_i = 0 \text{ on } B(\mathbf{0}, R_0) \setminus A(\mathbf{x}_i, 4R, R/2) \text{ and outside the part of the cone in } \mathbb{R}^3 \text{ centered at zero and passing through } S(\mathbf{0}, R_0) \cap B(\mathbf{x}_i, 4R) \text{ between } S(\mathbf{0}, R_0) \text{ and } S(\mathbf{0}, 2R_0). \quad (5.23)$$

Figure 2 illustrates the definition of ψ_i in the case $A(\mathbf{x}_i, 2R, R)$ is not entirely contained in $B(\mathbf{0}, R_0)$.

Similarly as in the previous section, we consider *optimal* coverings of $B(\mathbf{0}, R_0)$ by shells $\{A(\mathbf{x}_i, 2R, R)\}_{i=1,n}$ such that (5.20) is satisfied,

$$\left(\frac{R_0}{R}\right)^3 \leq n \leq K_1 \left(\frac{R_0}{R}\right)^3 \quad (5.24)$$

and

$$\text{any } \mathbf{x} \in B(\mathbf{0}, R_0) \text{ is covered by at most } K_2 \text{ shells } A(\mathbf{x}_i, 4R, R/2). \quad (5.25)$$

Introduce

$$\tilde{e}_{2R,R} = \frac{1}{n} \sum_{i=1}^n e_{\mathbf{x}_i, 2R, R}, \quad (5.26)$$

$$\tilde{E}_{2R,R} = \frac{1}{n} \sum_{i=1}^n E_{\mathbf{x}_i, 2R, R}, \quad (5.27)$$

and

$$\tilde{\Phi}_{2R,R} = \frac{1}{n} \sum_{i=1}^n \Phi_{\mathbf{x}_i, 2R, R}, \quad (5.28)$$

the ensemble averages of the time-averaged energy, enstrophy, and flux on the shells of thickness R corresponding to the covering $\{A(\mathbf{x}_i, 2R, R)\}_{i=1,n}$.

The ensemble average of the time-averaged modified flux on shells of thickness R is then defined by

$$\tilde{\Psi}_{2R,R} = \frac{1}{n} \sum_{i=1}^n \Psi_{\mathbf{x}_i, 2R, R} = \tilde{\Phi}_{2R,R} - \tilde{\Phi}_{2R,R}^\infty, \quad (5.29)$$

where

$$\tilde{\Phi}_{2R,R}^\infty = \frac{1}{n} \sum_{i=1}^n \frac{1}{T} \Phi_{\mathbf{x}_i, 2R, R}^\infty \quad (5.30)$$

is the ensemble average of the time-averaged anomalous fluxes through the shells of thickness R inside $B(\mathbf{0}, R_0)$.

An argument analogous to the one in Lemma 3.1 implies that if the covering of $B(\mathbf{0}, R_0)$ is optimal, then

$$\tilde{\Phi}_{2R,R}^\infty \leq K \frac{1}{T} \Phi_\infty. \quad (5.31)$$

Taking the time ensemble averages in (5.4) and applying the bounds (5.9–5.10), we arrive at

$$\tilde{\Psi}_{2R,R} \geq \nu \tilde{E}_{2R,R} - \nu \frac{C_0}{R^2} \tilde{e}_{2R,R}. \quad (5.32)$$

If the covering is optimal, i.e., if (5.20) and (5.24–5.25) hold, then

$$\tilde{E}_{2R,R} \geq \frac{1}{n} \tilde{E} \geq \frac{1}{K_1} \left(\frac{R}{R_0} \right)^3 \tilde{E} \quad (5.33)$$

and

$$\tilde{e}_{2R,R} \leq \frac{K_2}{n} \tilde{e} \leq K_2 \left(\frac{R}{R_0} \right)^3 \tilde{e}, \quad (5.34)$$

where

$$\tilde{E} = \frac{1}{T} \iint |\nabla \otimes \mathbf{u}|^2 \phi_0 \, d\mathbf{x} \, dt = R_0^3 E \quad (5.35)$$

is the time average of the localized enstrophy on $B(\mathbf{0}, R_0)$ and

$$\tilde{e} = \frac{1}{2} \frac{1}{T} \iint |\mathbf{u}|^2 \phi_0^{2\delta-1} \, d\mathbf{x} \, dt = R_0^3 e \quad (5.36)$$

is the time average of the localized energy on $B(\mathbf{0}, R_0)$ with ϕ_0 is defined by (3.10).

Let us note that

$$\tau_0 = \left(\frac{e}{E}\right)^{1/2} = \left(\frac{\tilde{e}}{\tilde{E}}\right)^{1/2}. \quad (5.37)$$

Utilizing (5.33), (5.34) and (5.37) in the inequality (5.32) gives

$$\tilde{\Psi}_{2R,R} \geq \frac{1}{K_1} \left(\frac{R}{R_0}\right)^3 \nu \tilde{E} \left(1 - C_0 K_1 K_2 \frac{\tau_0^2}{R^2}\right), \quad (5.38)$$

implying the following result.

Proposition 5.2. *Assume that the condition (4.13) holds for some $0 < \gamma < 1$. Then, for any R satisfying (4.15),*

$$c_{0,\gamma} \left(\frac{R}{R_0}\right)^3 \nu \tilde{E} \leq \tilde{\Psi}_{2R,R} \quad (5.39)$$

holds with c and $c_{0,\gamma}$ defined in (4.14) and (4.17).

Taking the time ensemble averages in the localized energy equality (5.4) again, this time looking for an upper bound, yields

$$\tilde{\Psi}_{2R,R} \leq \nu \tilde{E}_{2R,R} + \nu \frac{C_0}{R^2} \tilde{e}_{2R,R}.$$

If the covering $\{A(\mathbf{x}_i, 2R, R)\}_{i=1,n}$ of $B(\mathbf{0}, R_0)$ is optimal then, in addition to (5.34),

$$\tilde{E}_{2R,R} \leq \frac{K_2}{n} \tilde{E} \leq K_2 \left(\frac{R}{R_0}\right)^3 \tilde{E}; \quad (5.40)$$

hence,

$$\tilde{\Psi}_{2R,R} \leq \nu K_2 \left(\frac{R}{R_0}\right)^3 \tilde{E} + \nu K_2 \frac{C_0}{R^2} \left(\frac{R}{R_0}\right)^3 \tilde{e}.$$

As long as R is inside the inertial range delineated in (4.15),

$$\tilde{e} \leq c^2 \gamma^2 R^2 \tilde{E},$$

leading to

$$\tilde{\Psi}_{2R,R} \leq \nu K_2 \left(\frac{R}{R_0}\right)^3 \tilde{E} + \nu K_2 C_0 \left(\frac{R}{R_0}\right)^3 c^2 \gamma^2 \tilde{E} = c_{1,\gamma} \left(\frac{R}{R_0}\right)^3 \nu \tilde{E}.$$

Collecting the bounds on $\tilde{\Psi}_{2R,R}$ we establish the following.

Theorem 5.2. *Assume that the condition (4.13) holds for some $0 < \gamma < 1$. Then, for any R satisfying (4.15), the ensemble average of the time-averaged modified flux through the shells of thickness R , $\tilde{\Psi}_{2R,R}$, satisfies*

$$c_{0,\gamma} \left(\frac{R}{R_0}\right)^3 \nu \tilde{E} \leq \tilde{\Psi}_{2R,R} \leq c_{1,\gamma} \left(\frac{R}{R_0}\right)^3 \nu \tilde{E}, \quad (5.41)$$

where c , $c_{0,\gamma}$, and $c_{1,\gamma}$ are defined in (4.14) and (4.17) and the average is computed over a time interval $[0, 2T]$ with $T \geq R_0^2/\nu$ and determined by an optimal covering $\{A(\mathbf{x}_i, 2R, R)\}_{i=1,n}$ of $B(\mathbf{0}, R_0)$ (i.e. satisfying (5.20), (5.24), and (5.25)).

Note that if

$$\Psi_{2R,R} = \frac{1}{R^3} \tilde{\Psi}_{2R,R}$$

denotes the ensemble average of the time-space averaged modified flux through the shells of thickness R then, dividing (5.41) by R^3 , we obtain the following.

Corollary 5.1. *Under the conditions of the previous theorem,*

$$c_{0,\gamma} \nu E \leq \Psi_{2R,R} \leq c_{1,\gamma} \nu E. \quad (5.42)$$

Theorem 5.2 allows us to show locality of the time-averaged modified flux under the assumption (4.13). Indeed, the ensemble average of the time-averaged flux through the spheres of radius R satisfying (4.15) is

$$\tilde{\Psi}_R = R^3 \Psi_R.$$

According to Theorem 4.1,

$$c_{0,\gamma} \left(\frac{R}{R_0} \right)^3 \nu \tilde{E} \leq \tilde{\Psi}_R \leq c_{1,\gamma} \left(\frac{R}{R_0} \right)^3 \nu \tilde{E}.$$

On the other hand, the time ensemble average of the flux through the shells between spheres of radii R_2 and $2R_2$, according to Theorem 5.2 is

$$c_{0,\gamma} \left(\frac{R_2}{R_0} \right)^3 \nu \tilde{E} \leq \tilde{\Psi}_{2R_2,R_2} \leq c_{1,\gamma} \left(\frac{R_2}{R_0} \right)^3 \nu \tilde{E}.$$

Consequently,

$$\frac{c_{0,\gamma}}{c_{1,\gamma}} \left(\frac{R_2}{R} \right)^3 \leq \frac{\tilde{\Psi}_{2R_2,R_2}}{\tilde{\Psi}_R} \leq \frac{c_{1,\gamma}}{c_{0,\gamma}} \left(\frac{R_2}{R} \right)^3. \quad (5.43)$$

Thus, under the assumption (4.13), throughout the inertial range given by (4.15), the contribution of the shells at scales comparable to R is comparable to the total flux at scales R , the contribution of the shells at scales R_2 much smaller than R becomes negligible (ultraviolet locality) and the flux through the shells at scales R_2 much bigger than R becomes substantially bigger and thus essentially uncorrelated to the flux at scales R (infrared locality).

Moreover, if we choose $R_2 = 2^k R$ with k an integer, the relation (5.43) becomes

$$\frac{c_{0,\gamma}}{c_{1,\gamma}} 2^{3k} \leq \frac{\tilde{\Psi}_{2^{k+1}R, 2^k R}}{\tilde{\Psi}_R} \leq \frac{c_{1,\gamma}}{c_{0,\gamma}} 2^{3k}, \quad (5.44)$$

which implies that the aforementioned manifestations of locality propagate *exponentially* in the shell number k .

In contrast to (5.43), since $\tilde{e} = R_0^3 e$, $\tilde{E} = R_0^3 E$, $\tilde{\Psi}_{2R_2,R_2} = R_2^3 \Psi_{2R_2,R_2}$ and $\tilde{\Psi}_R = R^3 \Psi_R$,

$$\frac{c_{0,\gamma}}{c_{1,\gamma}} \leq \frac{\Psi_{2R_2,R_2}}{\Psi_R} \leq \frac{c_{1,\gamma}}{c_{0,\gamma}}, \quad (5.45)$$

i.e., the ensemble averages of the time-space averaged modified fluxes of the flows satisfying (4.13) are comparable throughout the scales involved in the inertial range (4.15) which is consistent with the universality of the cascade.

We conclude this section by noticing that the remarks similar to those at the end of Sect. 4 can be applied here. Namely we have the following.

Remark 5.4. Since $\tilde{\Phi}_{2R,R}^\infty \geq 0$, then all the lower bounds on modified fluxes hold for the usual fluxes $\tilde{\Phi}_{2R,R}$ and $\Phi_{2R,R}$; in particular, we have

$$\tilde{\Phi}_{2R,R} \geq c_1 \left(\frac{R}{R_0}\right)^3 \nu \tilde{E} \left(1 - c_2 \frac{\tau_0^2}{R^2}\right) \tag{5.46}$$

and, provided (4.13) and (4.15) hold,

$$\tilde{\Phi}_{2R,R} \geq c_{0,\gamma} \left(\frac{R}{R_0}\right)^3 \nu \tilde{E}. \tag{5.47}$$

Also, if a solution \mathbf{u} is such that (2.2) holds with equality and in particular, if the solution is regular, then all the estimates, including (5.41) and (5.42), hold for the usual averaged fluxes $\tilde{\Phi}_{2R,R}$ and $\Phi_{2R,R} = \tilde{\Phi}_{2R,R}/R^3$.

Remark 5.5. If the additional assumptions (5.22) and (5.23) for the test functions on the shells $A(\mathbf{x}_i, 2R, R)$ which are not contained entirely in $B(\mathbf{0}, R_0)$ are not imposed, then the lower bounds in (5.38) and (5.41) hold with \tilde{E} replaced by the time average of the non-localized in space enstrophy on $B(\mathbf{0}, R_0)$,

$$\tilde{E}' = \frac{1}{T} \int_0^{2T} \int_{B(\mathbf{0}, R_0)} |\nabla \otimes \mathbf{u}|^2 \eta \, d\mathbf{x} \, dt = R_0^3 E'.$$

This is the case because the estimate (5.33) gets replaced with

$$\tilde{E}_{2R,R} \geq \frac{1}{K_1} \left(\frac{R}{R_0}\right)^3 \tilde{E}'.$$

Also, the estimates (5.43) and (5.45) will contain the terms $E'/E (= \tilde{E}'/\tilde{E})$ in the lower and E/E' in the upper bounds.

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