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Localization and Geometric Depletion of Vortex-Stretching in the 3D NSE

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Abstract: Vortex-stretching and, consequently, the evolution of the vorticity is localized on an arbitrarily small space-time cylinder. This yields a complete localization of the geometric condition(s) for the regularity involving coherence of the vorticity direction. In particular, it implies the regularity of any geometrically constrained Leray solution independently of the type of the spatial domain or the boundary conditions.

1. Introduction

Numerical simulations and experiments reveal that the regions of intense vorticity self-organize in quasi-low-dimensional sparse coherent structures, e.g., vortex sheets and vortex tubes. An intriguing and challenging problem in mathematical analysis of the incompressible flows is to explain the influence of geometry of the regions of high vorticity magnitude on smoothness of the flow.

The geometric approach to studying smoothness/avoiding singularity formation in the 3D incompressible flows was pioneered in [Co] where Constantin derived a singular integral representation of the stretching factor in the evolution of the vorticity magnitude. The representation formula involves an explicit geometric kernel which is depleted by both local alignment and anti-alignment of the vorticity direction. Hence, local coherence of the vorticity direction, a purely geometric property, depletes the nonlinearity.

This type of geometric depletion of the nonlinearity was subsequently exploited in [CoFe]; the main result states that as long as the vorticity direction (in the region of intense vorticity) is Lipschitz-coherent, no blow-up can occur and the flow remains smooth. Following [CoFe], it was shown in [daVeigaBe1] that $\frac{1}{2}$-Hölder coherence suffices. In a related work [GrRu], a class of hybrid geometric-analytic conditions for avoiding singularity formation was obtained containing a purely geometric $\frac{1}{2}$-Hölder coherence and a purely analytic Beale-Kato-Majda condition (time-integrability of the L^∞-norm of the vorticity) as the endpoint cases.

It is possible to detect more structure, and, in particular, more cancelation properties in the integral form of the vortex-stretching term $\int_{\mathbb{R}^3} (\omega \cdot \nabla)u \cdot \omega = \int_{\mathbb{R}^3} \alpha |\omega|^2$ induced by Constantin's representation of the vortex stretching factor α. This was realized in [RuGr] and two types of results followed. One stating that concentration of the vorticity on small, sparsely populated vortex structures depletes the nonlinearity preventing a blow-up, and the other stating that a certain isotropy condition on curl $\omega = -\triangle u$ induces enough cancelations in the integral form of the vortex-stretching term to prevent singularity formation. It is worth noting that the Laplacian is a rotationally invariant operator; hence, this is essentially a condition on isotropy of the velocity field.

A different method of taking into account sparseness/thinness of the regions of intense vorticity is based on utilizing local-in-time spatial analyticity properties of the solutions to the 3D NSE via a plurisubharmonic measure maximum principle in $\mathbb{C}^3$ [Gr]. The result states that local existence of a thin direction–on a scale comparable to a localized vorticity version of the Kolmogorov dissipation scale– in the region of high vorticity magnitude suffices to control the L^∞-norm of the vorticity preventing the blow-up.

In all of the aforementioned results, the geometric conditions for preventing singularity formation, although being local in nature, were assumed uniformly on a time interval and uniformly in the region of intense vorticity throughout the spatial domain $\mathbb{R}^3$. In a recent work [GrZh], it was shown that it is possible to localize the conditions on coherence of the vorticity direction derived in [GrRu], and in particular, the purely geometric $\frac{1}{2}$-Hölder coherence, to an arbitrarily small space-time cylinder in $\mathbb{R}^3 \times (0, T)$.

At this point, a natural question arises: *Is it possible to obtain analogous results in the cases when the spatial domain is not the whole space* $\mathbb{R}^3$, e.g., in the case of standard non-slip (Dirichlet) boundary conditions on bounded and unbounded domains? In the case of the half-space $\mathbb{R}^3_+$ with the slip boundary conditions, $u_3 = 0$, $\frac{\partial u_j}{\partial x_3} = 0$, $1 \le j \le 2$, it was shown in [daVeiga1] that $\frac{1}{2}$-Hölder coherence of the vorticity direction suffices to prevent the singularity formation. In the case of the non-slip boundary conditions on smooth bounded domains, it was shown in [daVeiga2] that an analogous result is possible under a certain *assumption* on the *control of the normal derivative of the vorticity magnitude at the boundary*. A key step in the proofs was to obtain a version of the Biot-Savart law, i.e., to express the velocity in terms of the vorticity, for the type of the domain/boundary conditions in question; this lead to a suitable representation of the vortex-stretching term and ultimately to bounds on the enstrophy (the L^2-norm of the vorticity). In a very recent work [daVeigaBe2], the authors showed that $\frac{1}{2}$-Hölder coherence of the vorticity direction is a sufficient condition for the regularity in the case of the free boundary-type boundary conditions, $u \cdot n = 0$, $\omega \times n = 0$ (n is the exterior unit normal), on a bounded smooth domain. (In the case of the half-space, this type of boundary conditions reduces to the slip boundary conditions; hence, this result can be viewed as an extension of the work in [daVeiga1] to an arbitrary bounded smooth domain.)

In this paper, a localized representation formula for the vortex-stretching term is obtained leading to a control of the localized enstrophy for any Leray solution on an arbitrarily small space-time cylinder $Q_\delta(x_0, t_0) = B_\delta(x_0) \times (t_0 - \delta^2, t_0)$. This then yields a localization of the vorticity direction coherence conditions for the regularity to $Q_\delta(x_0, t_0)$. The proof merges the localization of the transport of the vorticity by the velocity previously obtained in [GrZh] with the newly obtained localization of vortex-stretching. (The localization of vortex-stretching presented in [GrZh] utilized the singular integral representation formula for the vortex-stretching factor α *over the whole space* $\mathbb{R}^3$ which was then split into small and large scales.)

This implies the regularity of *any* Leray solution with a $\frac{1}{2}$-Hölder coherent vorticity direction field *independently* of the type of the spatial domain or the boundary conditions.

2. Notation

The Navier-Stokes equations modeling the flow of a viscous incompressible fluid on a space-time domain $\Omega \times (0, T)$ (Ω an open subset of $\mathbb{R}^3$) read

$$u_t - \nu\triangle u + (u \cdot \nabla)u + \nabla p = 0, \tag{1}$$

supplemented with the incompressibility condition div$u = 0$, where the vector field $u = u(x,t)$ is the velocity of the fluid, the scalar field $p = p(x,t)$ is the pressure and the positive constant ν is the viscosity. (In what follows, the viscosity will be set to 1 – the results in the general case can be recovered by scaling.)

Taking the curl of (1) yields

$$\omega_t - \triangle\omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u\,, \tag{2}$$

where $\omega =$ curl u is the vorticity. The right-hand side of (2) , $(\omega \cdot \nabla)u$, is the vortex-stretching term which holds the key to understanding the phenomenon (or the lack thereof) of the singularity formation in the flow. (The other component of the nonlinearity, $(u \cdot \nabla)\omega$, is the part of the material derivative of the vorticity, i.e., of the transport of the vorticity by the velocity, and can be controlled.)

Let ξ denote the vorticity direction field, $\xi = \dfrac{\omega}{|\omega|}$, and S the symmetric part of the velocity gradient, $S = \dfrac{1}{2}\left(\nabla u + (\nabla u)^t\right)$. Define a key quantity α by $\alpha = S\xi \cdot \xi$. Then, a direct computation yields

$$(\omega \cdot \nabla)u \cdot \omega = S\omega \cdot \omega = \alpha|\omega|^2$$

and

$$(\partial_t + u \cdot \nabla - \triangle)|\omega|^2 + |\nabla\omega|^2 = \alpha|\omega|^2,$$

i.e., α represents the stretching factor in the evolution of the vorticity magnitude. Constantin (cf. [Co]) derived the following representation formula for α,

$$\alpha(x) = \frac{3}{4\pi} P.V. \int D\left(\hat{y}, \xi(x+y), \xi(x)\right) |\omega(x+y)| \frac{1}{|y|^3}\, dy,$$

where $\hat{y}$ is the unit vector in the y-direction and the *geometric* kernel D is defined by

$$D(e_1, e_2, e_3) = (e_1 \cdot e_3)\,(e_1 \cdot\ (e_2 \times e_3))$$

for arbitrary unit vectors e_1, e_2 and e_3. It is easily seen that

$$|D\left(\hat{y}, \xi(x+y), \xi(x)\right)| \le |\sin\varphi\,(\xi(x), \xi(x+y))\,|;$$

hence, coherence of the vorticity direction field softens up the singularity and depletes the nonlinearity (the vortex-stretching term).

For a point $(x_0, t_0) \in \Omega \times (0, T)$, denote by $Q_\delta(x_0, t_0)$ an open parabolic cylinder $B(x_0, \delta) \times (t_0 - \delta^2, t_0)$ contained in $\Omega \times (0, T)$.

3. Localization of Vortex-Stretching

As in [GrZh], let $\psi(x,t)=\phi(x)\eta(t)$ be a smooth cut-off function on $Q_{2r}(x_0,t_0)$ satisfying

$$\operatorname{supp}\phi\subset B(x_0,2r),\ \ \phi=1 \text{ on } B(x_0,r),\ \ \frac{|\nabla\phi|}{\phi^\rho}\le\frac{c}{r} \text{ for some } \rho\in(0,1),\ \ 0\le\phi\le 1$$

and

$$\operatorname{supp}\eta\subset(t_0-(2r)^2,t_0],\ \ \eta=1 \text{ on } [t_0-r^2,t_0],\ \ |\eta'|\le\frac{c}{r^2},\ \ 0\le\eta\le 1.$$

It was shown in [GrZh]–Sect. 2 that choosing any ρ, $\frac{1}{2}\le\rho<1$, leads to a suitable bound on the localized transport term $(u\cdot\nabla)\omega\cdot\psi^2\omega$. An additional restriction on ρ will transpire here in order to control the lower order terms in the localization of the vortex-stretching term.

The goal of this section is to obtain an *explicit localization formula* for the vortex stretching term $(\omega\cdot\nabla)u\cdot\omega$ on $Q_{2r}(x_0,t_0)$. The computation will be uniform in time; thus the time variable t will be omitted.

Let x be in $B(x_0,2r)$, and consider

$$\phi^2(x)(\omega\cdot\nabla)u\cdot\omega\,(x)=\phi(x)\,\frac{\partial}{\partial x_i}u_j(x)\,\phi(x)\,\omega_i(x)\,\omega_j(x).$$

First write ϕu_j as

$$\begin{aligned}
\phi(x)u_j(x)&=c\int_{B(x_0,2r)}\frac{1}{|x-y|}\triangle\left(\phi u_j\right)\,dy\\
&=c\int_{B(x_0,2r)}\frac{1}{|x-y|}\phi\,\triangle u_j\,dy\\
&\quad+c\int_{B(x_0,2r)}\frac{1}{|x-y|}\left(2\nabla\phi\cdot\nabla u_j+\triangle\phi\,u_j\right)\,dy\\
&=c\int_{B(x_0,2r)}\frac{1}{|x-y|}\phi\,(\operatorname{curl}\omega)_j\,dy\\
&\quad+c\int_{B(x_0,2r)}\frac{1}{|x-y|}\left(2\nabla\phi\cdot\nabla u_j+\triangle\phi\,u_j\right)\,dy\\
&=I_1+I_2\,,
\end{aligned}$$

and note that the terms in I_2 are the lower order terms with respect to I_1.

Using the Levi-Civita symbol ϵ_{jkl}, I_1 can be written in the following way,

$$\begin{aligned}
I_1&=c\int_{B(x_0,2r)}\frac{1}{|x-y|}\phi\,\epsilon_{jkl}\,\frac{\partial}{\partial y_k}\omega_l\,dy\\
&=-c\int_{B(x_0,2r)}\epsilon_{jkl}\,\frac{\partial}{\partial y_k}\frac{1}{|x-y|}\,\phi\,\omega_l\,dy+-c\int_{B(x_0,2r)}\epsilon_{jkl}\,\frac{1}{|x-y|}\,\frac{\partial}{\partial y_k}\phi\,\omega_l\,dy\\
&=J_1+J_2.
\end{aligned}$$

Hence,

$$\phi u_j = J_1 + J_2 + I_2, \tag{3}$$

where J_2 and I_2 are the lower order terms with respect to J_1. Differentiating the localized Biot-Savart law (3) yields

$$\frac{\partial}{\partial x_i}(\phi u_j)(x) = -c\, P.V. \int_{B(x_0,2r)} \epsilon_{jkl} \frac{\partial^2}{\partial x_i \partial y_k} \frac{1}{|x-y|}\, \phi\, \omega_l\, dy + \left(\frac{\partial}{\partial x_i} J_2 + \frac{\partial}{\partial x_i} I_2\right).$$

Writing $\phi \dfrac{\partial}{\partial x_i} u_j = \dfrac{\partial}{\partial x_i}(\phi u_j) - \dfrac{\partial}{\partial x_i}\phi\, u_j$, this implies our localization formula,

$$\begin{aligned}
&\phi^2(x)(\omega \cdot \nabla)u \cdot \omega\,(x) \\
&= \phi(x)\, \frac{\partial}{\partial x_i} u_j(x)\, \phi(x)\, \omega_i(x)\, \omega_j(x) \\
&= -c\, P.V. \int_{B(x_0,2r)} \epsilon_{jkl} \frac{\partial^2}{\partial x_i \partial y_k} \frac{1}{|x-y|}\, \phi\, \omega_l\, dy\, \phi(x)\, \omega_i(x)\, \omega_j(x) + \text{ LOT} \\
&= -c\, P.V. \int_{B(x_0,2r)} (\omega(x) \times \omega(y)) \cdot G_\omega(x,y)\, \phi(y)\, \phi(x)\, dy + \text{ LOT} \\
&= \text{VST}_{loc} + \text{ LOT},
\end{aligned} \tag{4}$$

where

$$(G_\omega(x,y))_k = \frac{\partial^2}{\partial x_i \partial y_k} \frac{1}{|x-y|}\, \omega_i(x)$$

and LOT denotes the terms that are the lower order terms with respect to VST_{loc} in the sense they are either lower order for at least one order of the differentiation or/and less singular for at least one power of $|x-y|$.

4. Control of the Localized Enstrophy

Let u be a Leray solution on $\Omega \times (0,T)$, i.e., a weak (distributional) solution satisfying $u \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$. Fix a point (x_0,t_0) and let $R > 0$ be such that $Q_{2R}(x_0,t_0) \subset \Omega \times (0,T)$. For simplicity of the exposition, we will assume that u is smooth on an open parabolic cylinder $Q_{2R}(x_0,t_0)$ and obtain bounds on the enstrophy localized to $B(x_0,R)$ *uniformly* in t in $(t_0 - R^2, t_0)$.

Let $r \le R$. A direct calculation shows (cf. [GrZh] – Sect. 2) that multiplying the vorticity equations (2) by $\psi^2\omega$ and integrating over $Q_{2r}^s = B(x_0,2r) \times (t_0 - (2r)^2, s)$, for a fixed s in $(t_0 - (2r)^2, t_0)$, yields

$$\begin{aligned}
&\frac{1}{2}\int_{B(x_0,2r)} \phi^2(x)|\omega|^2(x,s)\, dx + \int_{Q_{2r}^s} |\nabla(\psi\omega)|^2\, dxdt \\
&\le \int_{Q_{2r}} \left(|\eta||\partial_t \eta| + |\nabla\psi|^2\right) |\omega|^2\, dxdt \\
&\quad + \left|\int_{Q_{2r}^s} (u \cdot \nabla)\omega \cdot \psi^2\omega\, dxdt\right| + \left|\int_{Q_{2r}^s} (\omega \cdot \nabla)u \cdot \psi^2\omega\, dxdt\right| \\
&= T_1 + T_2 + T_3.
\end{aligned}$$

It is plain that

$$T_1 \leq c(r) \int_{Q_{2r}} |\omega|^2 \, dx dt, \tag{5}$$

and the following bound on the localized transport term T_2 was derived in [GrZh] – Sect. 2,

$$T_2 \leq \frac{1}{2} \int_{Q_{2r}} |\nabla(\psi\omega)|^2 \, dx dt + c(r) \int_{Q_{2r}} |\omega|^2 \, dx dt. \tag{6}$$

To estimate the localized vortex-stretching term T_3, we will utilize the representation formula (4). (As mentioned in the introduction, the estimate in [GrZh]–Sect. 3 made use of the singular integral representation formula of the vortex-stretching factor α over the whole space $\mathbb{R}^3$ which was then separated in small and large scales.)

Bringing back to life the time variable t and multiplying both sides of (4) by η^2 implies

$$T_3 \leq c \int_{Q_{2r}^s} P.V. \int_{B(x_0,2r)} \frac{1}{|x-y|^3} \, \psi(y,t) \, |\omega|(y,t) \, \psi(x,t) \, |\omega|^2(x,t) \, dy \, dx dt \tag{7}$$
$$+ \text{ the lower order terms,}$$

where each lower order term is at least for one order of differentiation or/and at least one power of $|x-y|$ less singular than the principal term.

This bound is insufficient to close the estimate on the localized enstrophy. A geometric structure of the leading term in (4) will be exploited in the following section to show that $\frac{1}{2}$-Hölder coherence of the vorticity direction depletes the nonlinearity preventing a possible singularity formation.

5. Localization of the Coherence of the Vorticity Direction Field Condition for the Regularity

Theorem 1. *Let $\Omega \subseteq \mathbb{R}^3$ be open, and u a Leray solution on the space-time domain $\Omega \times (0,T)$ for some $T > 0$. Fix a point (x_0,t_0) in $\Omega \times (0,T)$, and let $0 < R < 1$ be such that the open parabolic cylinder $Q_{2R}(x_0,t_0) = B(x_0,2R) \times (t_0 - (2R)^2, t_0)$ is contained in $\Omega \times (0,T)$.*

Suppose that u is smooth on $Q_{2R}(x_0,t_0)$ and that there exist two positive constants K, M such that the following coherence condition holds,

$$|\sin\varphi\,(\xi(x,t),\xi(y,t))\,| \leq K|x-y|^{\frac{1}{2}}$$

for all (x,t), (y,t) in $Q_{2R} \cap \{|\omega| > M\}$.

Then the localized enstrophy remains uniformly bounded up to $t = t_0$, i.e.,

$$\sup_{t\in(t_0-R^2,t_0)} \int_{B(x_0,R)} |\omega|^2(x,t)\,dx < \infty.$$

Proof. It remains to estimate T_3. First, separate the contribution of the region of the low vorticity magnitude from the contribution of the region of the high vorticity magnitude,

$$T_3 \leq \left| \int_{Q_{2r}^s \cap \{|\omega|<M\}} (\omega\cdot\nabla)u \cdot \psi^2 \omega \, dx dt \right|$$
$$+ \left| \int_{Q_{2r}^s \cap \{|\omega|>M\}} (\omega\cdot\nabla)u \cdot \psi^2 \omega \, dx dt \right| = T_3^l + T_3^h.$$

The first term, T_3^l, is bounded by $c\int_{Q_{2r}} |\nabla u|^2\,dxdt$. Utilizing $\frac{1}{2}$-Hölder coherence of the vorticity direction and (4), the following estimate transpires,

$$T_3^h \le c\int_{Q_{2r}^s}\int_{B(x_0,2r)} \frac{1}{|x-y|^{\frac{5}{2}}}\,\psi(y,t)\,|\omega|(y,t)\,\psi(x,t)\,|\omega|(x,t)\,|\omega|(x,t)\,dy\,dxdt$$
$$+\text{ the lower order terms} = I + I_{LOT}. \tag{8}$$

For the leading term in the above estimate, I, we apply Hölder (in x) with the exponents 4, 4 and 2, and then the weak Young to the first factor, leading to

$$\begin{aligned} I &\le \int_{t_0-(2r)^2}^{s} \|\psi\omega(t)\|_{L^{\frac{12}{5}}(\mathbb{R}^3)}\,\|\psi\omega(t)\|_{L^4(\mathbb{R}^3)}\,\|\omega(t)\|_{L^2(B(x_0,2r))}\,dt \\ &\le \int_{t_0-(2r)^2}^{t_0} \|\psi\omega(t)\|_{L^{\frac{12}{5}}(\mathbb{R}^3)}\,\|\psi\omega(t)\|_{L^4(\mathbb{R}^3)}\,\|\omega(t)\|_{L^2(B(x_0,2r))}\,dt. \end{aligned}$$

This followed by interpolating the first two L^p-norms yields

$$\begin{aligned} I &\le \int_{t_0-(2r)^2}^{t_0} \|\nabla(\psi\omega)(t)\|_{L^2(\mathbb{R}^3)}\,\|\psi\omega(t)\|_{L^2(\mathbb{R}^3)}\,\|\omega(t)\|_{L^2(B(x_0,2r))}\,dt \\ &= \int_{t_0-(2r)^2}^{t_0} \|\nabla(\psi\omega)(t)\|_{L^2(B(x_0,2r))}\,\|\psi\omega(t)\|_{L^2(B(x_0,2r))}\,\|\omega(t)\|_{L^2(B(x_0,2r))}\,dt \\ &\le \sup_{t\in(t_0-(2r)^2,t_0)} \|\phi\omega(t)\|_{L^2(B(x_0,2r))}\,\|\nabla(\psi\omega)\|_{L^2(Q_{2r})}\,\|\omega\|_{L^2(Q_{2r})} \\ &\le \|\omega\|_{L^2(Q_{2r})}\left(\frac{1}{2}\sup_{t\in(t_0-(2r)^2,t_0)} \|\phi\omega(t)\|^2_{L^2(B(x_0,2r))} + \|\nabla(\psi\omega)\|^2_{L^2(Q_{2r})}\right). \end{aligned}$$

Collecting the estimates on T_1, T_2, T_3^l and T_3^h, we arrive at the following bound,

$$\begin{aligned} &\frac{1}{2}\int_{B(x_0,2r)} \phi^2(x)|\omega|^2(x,s)\,dx + \int_{Q_{2r}^s} |\nabla(\psi\omega)|^2\,dxdt \\ &\quad\le c(r)\int_{Q_{2r}} |\omega|^2\,dxdt \\ &\qquad+\frac{1}{2}\int_{Q_{2r}} |\nabla(\psi\omega)|^2\,dxdt + c(r)\int_{Q_{2r}} |\omega|^2\,dxdt \\ &\qquad+c\int_{Q_{2r}} |\nabla u|^2\,dxdt \\ &\qquad+\|\omega\|_{L^2(Q_{2r})}\left(\frac{1}{2}\sup_{t\in(t_0-(2r)^2,t_0)} \|\phi\omega(t)\|^2_{L^2(B(x_0,2r))} + \|\nabla(\psi\omega)\|^2_{L^2(Q_{2r})}\right) \\ &\qquad+I_{LOT} \end{aligned}$$

for all s in $(t_0 - (2r)^2, t_0)$. Hence,

$$
\begin{aligned}
&\frac{1}{2} \sup_{t \in (t_0-(2r)^2, t_0)} \|\phi\omega(t)\|^2_{L^2(B(x_0,2r))} + \|\nabla(\psi\omega)\|^2_{L^2(Q_{2r})} \\
&\le c(r) \int_{Q_{2r}} |\nabla u|^2 \, dx dt \\
&\quad + 2\, \|\nabla u\|_{L^2(Q_{2r})} \left(\frac{1}{2} \sup_{t \in (t_0-(2r)^2, t_0)} \|\phi\omega(t)\|^2_{L^2(B(x_0,2r))} + \|\nabla(\psi\omega)\|^2_{L^2(Q_{2r})} \right) \\
&\quad + I_{LOT}.
\end{aligned}
$$

Since u is a Leray solution, the first term in the estimate is finite. Moreover, since $\|\nabla u\|_{L^2(Q_{2r})} \to 0,\ r \to 0$, there exist $\delta > 0$ such that $\|\nabla u\|_{L^2(Q_{2\rho})} \le \frac{1}{4}$ for all $\rho \le \delta$. If $r \le \delta$, the second term is absorbed. If $r > \delta$, a desired bound is obtained by covering $B_r(x_0, t_0)$ with finitely many balls $B_\delta(z_0, t_0)$ and redoing the proof on each cylinder $Q_{2\delta}(z_0, t_0)$.

At the end, let us briefly address the lower order terms, I_{LOT}. Consider a highest order lower order term I^h_{LOT},

$$
I^h_{LOT} = \int_{Q_{2r}} \int_{B(x_0,2r)} \frac{1}{|x-y|^2} \, |\nabla\psi|(y,t)\, |\nabla u|(y,t)\, \psi(x,t)\, |\omega|^2(x,t)\, dy\, dxdt.
$$

Let $\frac{1}{2} \le \rho < 1$. By construction of the cut-off ϕ, $|\nabla\phi(y)| \le c_\rho \frac{1}{r} \phi^\rho(y)$, i.e., we can estimate the gradient of the cut-off by the ρ-power of the cut-off. A problem here is that the cut-off is in the wrong variable (there is no use localizing $|\nabla u|$). To switch localization from y to x use the Mean Value Theorem to write

$$
\phi^\rho(y) = \phi^\rho(x) + \rho \frac{\nabla\phi(z)}{\phi^{1-\rho}(z)} \cdot (y - x).
$$

Since $\rho \ge \frac{1}{2}$,

$$
|\nabla\phi(y)| \le c_1(\rho) \frac{1}{r} \phi^\rho(x) + c_2(\rho) \frac{1}{r^2} |x - y|.
$$

This leads to the following bound,

$$
\begin{aligned}
&I^h_{LOT} \\
&\le c_1(\rho) \int_{Q_{2r}} \int_{B(x_0,2r)} \frac{1}{|x-y|^2} \, |\nabla u|(y,t) \left(\frac{1}{r} |\omega|^{1-\rho}(x,t) \right) |\psi\omega|^{1+\rho}(x,t)\, dy\, dxdt \\
&\quad + c_2(\rho) \frac{1}{r^2} \int_{Q_{2r}} \int_{B(x_0,2r)} \frac{1}{|x-y|} \, |\nabla u|(y,t)\, |\omega|(x,t)\, |\psi\omega|(x,t)\, dy\, dxdt \\
&= A + B.
\end{aligned}
$$

For the first term, notice that were the limit case $\rho = 1$ possible, the critical kernel would be $\frac{1}{|x-y|^{\frac{5}{2}}}$. Namely, Hölder (in x) with the exponents 3 and $\frac{3}{2}$ followed by the weak Young would yield the bound

$$\int_{t_0(2r)^2}^{t_0} \|\nabla u(t)\|_{L^2(B(x_0,2r))} \, \|\psi\omega(t)\|^2_{L^3(B(x_0,2r))} \, dt;$$

interpolating the second norm, this is easily bounded by

$$\|\nabla u\|_{L^2(Q_{2r})} \left(\frac{1}{2} \sup_{t\in(t_0-(2r)^2,t_0)} \|\phi\omega(t)\|^2_{L^2(B(x_0,2r))} + \|\nabla(\psi\omega)\|^2_{L^2(Q_{2r})} \right).$$

Since there is a gap between our kernel and the critical kernel, it is possible to choose ρ sufficiently close to 1 in order to separate out $\left(\frac{1}{r}|\omega|^{1-\rho}\right)$ factor and bound its contribution by a term depending only on r and $\|\nabla u\|_{L^2(Q_{2r})}$.

For the second term, a crude estimate (Hölder (in x) with the exponents ∞, 2 and 2 followed by Cauchy-Schwartz in y) yields

$$B \le c(\rho, r) \int_{t_0-(2r)^2}^{t_0} \|\nabla u(t)\|_{L^2(B(x_0,2r))} \, \|\omega(t)\|_{L^2(B(x_0,2r))} \, \|\psi\omega(t)\|_{L^2(B(x_0,2r))} \, dt,$$

and this is bounded by

$$\epsilon \sup_{t\in(t_0-(2r)^2,t_0)} \|\phi\omega(t)\|^2_{L^2(B(x_0,2r))} + c(\rho, r, \epsilon)\|\nabla u\|^4_{L^2(Q_{2r})}$$

for any $\epsilon > 0$. $\square$

Remark 1. The proof can be modified leading to localization of the hybrid geometric-analytic conditions for the regularity derived in [GrRu].

Remark 2. The theorem implies interior regularity of any Leray solution possessing $\frac{1}{2}$-Hölder coherent vorticity direction field independently of the type of the domain or the boundary conditions. This, in particular, provides a partially positive answer to a question raised in [daVeiga2] whether in the case of the non-slip boundary conditions on a bounded domain, the coherence of the vorticity direction field alone, without any extra assumptions, implies the regularity.

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