Commun. Math. Phys. 247, 601–611 (2004)
Digital Object Identifier (DOI) 10.1007/s00220-004-1072-0

Communications in
**Mathematical Physics**

# On Depletion of the Vortex-Stretching Term in the 3D Navier-Stokes Equations

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Received: 18 January 2003 / Accepted: 20 November 2003
Published online: 30 April 2004 – 

**Abstract:** Certain new cancellation properties in the vortex-stretching term are detected leading to new geometric criteria for preventing finite-time blow-up in the 3D Navier-Stokes equations.

## 1. Introduction

There is an extensive literature on formulating sufficient conditions for regularity of solutions of the 3D Navier-Stokes equations. The most classical ones are the space-time integrability conditions on the velocity $u$ [P]

$$\|u(\cdot)\|_q^{2q/(q-3)} \in L^1(0, T)$$

for some $q \in (3, \infty]$. A localized version was given in [S] and a weak $L^3$-version can be found in [Ko]. There are analogous space-time integrability conditions for the vorticity $\omega = \operatorname{curl} u$ – in particular the Beale-Kato-Majda

$$\|\omega(\cdot)\|_\infty \in L^1(0, T)$$

(originally derived for the 3D Euler equations [BKM]). This condition has been recently weakened [KoT] to

$$\|\omega(\cdot)\|_{BMO} \in L^1(0, T)$$

via a sharp $BMO$ bilinear estimate.

Constantin [C2] discovered an integral representation of the stretching factor in the evolution of the vorticity magnitude $\alpha$ revealing that local alignment of the vorticity directions, a geometric condition, depletes $\alpha$. (It should be noted that coherent vortex structures, e.g., vortex sheets and vortex tubes, exhibit local alignment of the vorticity directions.) This phenomenon was then exploited in [CF] where it was shown that as long as the vorticity direction satisfies a Lipschitz-like regularity no blow-up can occur. More recently the Lipschitz condition was scaled down to a $1/2$-Hölder-like regularity [BdVB]. In a work just completed, the authors formulated a more general sufficient condition for the regularity including the $1/2$-Hölder condition and the Beale-Kato-Majda condition as the 'end-point cases' [GR].

A different geometric condition controlling the growth of the vorticity magnitude was presented in [G] – essentially, local existence of a sparse direction in the regions of high vorticity magnitude on the scales comparable to a localized vorticity version of the Kolmogorov dissipation scale. The proof relied on certain estimates on the complexified solutions in $L^\infty$ and a plurisubharmonic measure maximum principle in $\mathbb{C}^3$, and a minimal scale was derived from a lower bound on the uniform radius of spatial analyticity.

In this paper we study a representation of the $L^2$-product of the vortex stretching term with $\omega$ induced by the aforementioned integral representation of $\alpha$. Rewriting the integral in a suitable form and utilizing some cancellation properties we first show that small scales in a convolution integral are harmless. More precisely, the convolution integral restricted to the balls $B_r$, where $r \le r^*$ is bounded by the (viscous) positive term appearing in the evolution of enstrophy (it is quite intriguing that a minimal scale $r^*$ *coincides* with the minimal scale that appeared in [G] although the techniques are completely different). Since the convolution integral restricted to large scales can be controlled via known *a priori* estimates on $\omega$ we immediately see that if the regions of high fluid activity are comprised of sparsely populated small-scale structures no blow-up can occur. Next, more cancellation properties are detected leading to a proof that a certain *isotropy* condition effectively controls the evolution of enstrophy preventing finite-time blow-up. This is interesting since the previous conditions – regularity, i.e. the alignment of the vorticity directions – are essentially *anisotropic*.

## 2. Preliminaries

We consider the vorticity formulation of the 3D NSE,

$$\omega_t - \nu\triangle\omega + (u\cdot\nabla)\omega = (\omega\cdot\nabla)u, \tag{1}$$

where $u$ is the velocity of the fluid, $\operatorname{div} u = 0$, $\omega = \operatorname{curl} u$ is the vorticity and $\nu > 0$ is the viscosity. The spatial domain $\Omega$ will be the whole space $\mathbb{R}^3$. The right-hand-side term $(\omega\cdot\nabla)u$ is the vortex-stretching term and is absent in the 2D case. The other nonlinear term is only virtually there – since $u$ is divergence free, it vanishes when multiplied with $\omega|\omega|^{2p-2}$ in any $L^p$, $p \ge 1$.

Denote by $\xi$ the vorticity direction, $\xi = \frac{\omega}{|\omega|}$. The strain tensor (or the deformation tensor) $S$ is given by $S = \frac{1}{2}(\nabla u + \nabla u^T)$, and a key quantity $\alpha$ is defined by $\alpha = S\xi\cdot\xi$. A direct calculation shows that the vortex-stretching term $(\omega\cdot\nabla)u$ is equal to $S\omega$ and that $S\omega\cdot\omega = \alpha|\omega|^2$ (and more generally $S\omega\cdot\omega|\omega|^{2p-2} = \alpha|\omega|^{2p}$). Hence $\alpha$ effectively controls the evolution of the enstrophy $\|\omega\|_2^2$ (and more generally any $L^q$-norm of $\omega$, $q \ge 2$).

Constantin [C2] derived an integral representation of $\alpha$ which explicitly revealed *geometric* depletion of $\alpha$. Denoting by $\hat{y}$ the unit vector in the $y$-direction,

$$\alpha(x) = \frac{3}{4\pi} P.V. \int D(\hat{y}, \xi(x+y), \xi(x))|\omega(x+y)|\frac{dy}{|y|^3}, \tag{2}$$

where the geometric factor $D$ is proportional to the volume spanned by the unit vectors $\hat{y}$, $\xi(x+y)$, and $\xi(x)$. More precisely,

$$D(e_1, e_2, e_3) = (e_1 \cdot e_3)\mathrm{Det}(e_1, e_2, e_3) \tag{3}$$

for any triplet of unit vectors $e_1, e_2$, and $e_3$. It is easily seen that $|D| \leq |\sin\varphi| \leq |\xi(x+y) - \xi(x)|$ ($\varphi$ denotes the angle between $\xi(x+y)$ and $\xi(x)$) – hence regularity of $\xi$, i.e., local alignment of the vorticity directions, will deplete the $\alpha$-singularity.

Since geometric conditions will be assumed only in the regions of high fluid activity we introduce notation for the appropriate super-level sets of $\omega$ and $\operatorname{curl}\omega$. For $M > 0$, $t > 0$ define

$$\Omega_t(M) = \{x \in \Omega : |\omega(x,t)| \geq M\}$$

and

$$\Psi_t(M) = \{x \in \Omega : |\operatorname{curl}\omega(x,t)| \geq M\}.$$

Utilizing the geometric depletion of $\alpha$, Constantin and Fefferman [CF] proved that as long as the vorticity direction satisfies Lipschitz-like regularity no blow-up can occur.

**Theorem 1 ([CF]).** *Assume that there exist constants $c, M > 0$ such that*

$$|\sin\varphi| \leq c|y|$$

*for all $x \in \Omega_t(M)$, $t \in (0,T)$. Then $\lim_{t\uparrow T} \|\omega(t)\|_2 < \infty$.*

Following the approach of [CF], Beirao da Veiga and Berselli [BdVB] showed that 1/2-Hölder-like regularity of $\xi$ suffices.

**Theorem 2 ([BdVB]).** *Assume that there exist constants $c, M > 0$ such that*

$$|\sin\varphi| \leq c|y|^{1/2}$$

*for all $x \in \Omega_t(M)$, $t \in (0,T)$. Then $\lim_{t\uparrow T} \|\omega(t)\|_2 < \infty$.*

The above theorems were proved controlling the enstrophy via the geometric depletion of $\alpha$. The following theorem was obtained controlling $L^q$-norms ($q \geq 2$) via the geometric depletion of $\alpha$. It can be viewed as an interpolation result between the 1/2-Hölder condition and the Beale-Kato-Majda condition.

**Theorem 3 ([GR]).** *Assume that there exist absolute constants $c, M > 0$ such that for some $q \geq 2$,*

*i)* $\|\omega\|_q^{q/(q-1)} \in L^1(0,T)$,
*ii)* $|\sin\varphi| \leq c|y|^{1/q}$ *for all $x \in \Omega_t(M)$, $t \in (0,T)$.*

*Then $\lim_{t\uparrow T} \|\omega(t)\|_q < \infty$.*

## 3. Small Scales and Large Scales

In this section we obtain an explicit (and *rigorous*) manifestation of a basic principle of turbulence: viscosity dominates the nonlinearity on small scales. A minimal scale *coincides* with the one derived in [G] and is a localized $\omega$-version of the Kolmogorov dissipation scale.

Throughout the paper we consider solutions smooth (regular) on an open interval $(0, T)$ and are interested in a possible loss of regularity as $t \uparrow T$. Local-in-time existence of smooth (in fact analytic in both space and time) solutions is guaranteed by, e.g., $\omega_0 = \omega(0) \in L^2$. In addition, assuming $u_0 = u(0) \in L^2$ and $\omega_0 \in L^1$ provides a number of useful *a priori* estimates [C1, FGT]: $\|\omega(\cdot)\|_1 \in L^\infty(0, \tau)$, $\|\omega(\cdot)\|_\infty^{1/2} \in L^1(0, \tau)$ and $\|\nabla\omega(\cdot)\|_{4/(3+\epsilon)}^{4/(3+\epsilon)} \in L^1(0, \tau)$ (any $0 < \epsilon \le \frac{1}{2}$) for any $\tau > 0$.

Multiplying (1) by $\omega$ in $L^2$ yields the following expression for the evolution of enstrophy

$$\frac{1}{2}\frac{d}{dt}\int |\omega|^2\,dx + \nu\int |\nabla\omega|^2\,dx = \int S\omega\cdot\omega\,dx = \int \alpha|\omega|^2\,dx. \tag{4}$$

Utilizing the integral representation of $\alpha$ (2), the nonlinear term can be written as

$$I \equiv \iint (\omega(x)\cdot\hat{y})(\omega(x+y)\times\omega(x)\cdot\hat{y})\frac{1}{|y|^3}\,dy\,dx. \tag{5}$$

(The $y$-integral is to be understood in the $P.V.$-sense.)

First, we derive some estimates on $I$ restricted to small and large $y$-scales. For $0 < r < \infty$ and $0 < r_1 < r_2 < \infty$ define $I_r$, $I_{r^C}$ and $I_{r_1,r_2}$ by

$$I_r = \int\int_{|y|\le r} (\omega(x)\cdot\hat{y})(\omega(x+y)\times\omega(x)\cdot\hat{y})\frac{1}{|y|^3}\,dy\,dx,$$
$$I_{r^C} = \int\int_{|y|\ge r} (\omega(x)\cdot\hat{y})(\omega(x+y)\times\omega(x)\cdot\hat{y})\frac{1}{|y|^3}\,dy\,dx,$$

and

$$I_{r_1,r_2} = \int\int_{r_1\le|y|\le r_2} (\omega(x)\cdot\hat{y})(\omega(x+y)\times\omega(x)\cdot\hat{y})\frac{1}{|y|^3}\,dy\,dx.$$

*Remark 4.* In the following lemmas and propositions $\omega$ is a smooth vector field for which the manipulations performed in a proof are legal and yield finite quantities.

**Proposition 5 (Control on small scales).** *Let* $\nu > 0$ *and* $r \le r^* \equiv \frac{1}{2\sqrt{\pi}}\frac{\nu^{1/2}}{\|\omega\|_\infty^{1/2}}$. *Then*

$$|I_r| \le \frac{\nu}{4}\int |\nabla\omega|^2\,dx.$$

*Proof.* Using the properties of the cross product and changing some variables we obtain

$$\begin{aligned} I_r &= \int\int_{|y|\le r} (\omega(x)\cdot\hat{y})(\omega(x+y)\times\omega(x)\cdot\hat{y})\frac{1}{|y|^3}\,dy\,dx \\ &= \frac{1}{2}\int\int_{|y|\le r} ((\omega(x)-\omega(x+y))\cdot\hat{y})(\omega(x+y) \\ &\quad \times(\omega(x)-\omega(x+y))\cdot\hat{y})\frac{1}{|y|^3}\,dy\,dx. \end{aligned} \tag{6}$$

By the Fundamental Theorem of Calculus

$$\omega(x) - \omega(x+y) = -|y| \int_0^1 \nabla\omega(x+sy)\hat{y}\, ds,$$

and thus

$$|I_r| = \frac{1}{2}\left| \int \int_0^1 ds_1 \int_0^1 ds_2 \int_{|y|\le r} (\nabla\omega(x+s_1y)\hat{y}\cdot\hat{y})(\omega(x+y)\times\nabla\omega(x+s_2y)\hat{y}\cdot\hat{y})\frac{1}{|y|}dydx \right|$$
$$\le \frac{1}{2}\int \int_0^1 ds_1 \int_{|y|\le r} |\nabla\omega(x+s_1y)|^2 \|\omega\|_\infty \frac{1}{|y|}dydx \le \pi r^2 \|\omega\|_\infty \int |\nabla\omega(z)|^2 dz. \quad (7)$$

□

The nonlinearity restricted to large scales can be bounded in terms of *a priori* estimates on $\omega$. We will use the following simple inequality.

**Proposition 6 (Control on large scales).** *Let* $R > 0$. *Then*

$$|I_{R^C}| \le \frac{1}{R^3}\|\omega\|_1 \int |\omega|^2\, dx.$$

*Proof.*

$$|I_{R^C}| \le \frac{1}{R^3}\int |\omega(x)|^2 \int |\omega(x+y)|\, dy\, dx = \frac{1}{R^3}\|\omega\|_1 \int |\omega|^2\, dx.$$

□

Henceforth, for two positive functions $f$ and $g$ on $(0,T)$, $f(t) \sim g(t)$ means there is a constant $c > 1$ such that $\frac{1}{c}g(t) \le f(t) \le cg(t)$ for all $t$ in $(0,T)$.

**Theorem 7.** *Let $\omega$ be a smooth solution of (1) on $(0,T)$ for some $T > 0$ corresponding to the initial data satisfying $u_0 \in L^2$ and $\omega_0 \in L^1 \cap L^2$. Let $r^*(t) = \frac{\nu^{1/2}}{2\pi^{1/2}\|\omega(t)\|_\infty^{1/2}}$, $R^*(t) \sim \frac{1}{\|\omega(t)\|_\infty^{1/6}}$ and $M^*(t)$ such that $M^*(t)(\log M^*(t))^3 \sim \|\omega(t)\|_\infty^{1/2}$, and assume that for every $t$ in $(0,T)$ the super-level set $\Omega_t(M^*(t))$ is a union of small-scale vortex structures $\{V_\beta(t)\}$ ($\beta$ in some index set) satisfying the following two properties:*

*i)* $\mathrm{diam} V_\beta(t) \le r^*(t)$,
*ii)* $\mathrm{dist}(V_\beta(t), V_\gamma(t)) \ge R^*(t)$, $\beta \ne \gamma$.

*Then $\lim_{t\uparrow T} \|\omega(t)\|_2 < \infty$, i.e. $T$ is not a singular time, and thus $\omega$ can be extended smoothly beyond $T$.*

*Remark 8.* The extension $\widetilde{\omega}$ is actually analytic (both in space and time). Namely, $\|\widetilde{\omega}(T)\|_2 < \infty$ and since $\|\widetilde{\omega}(\cdot)\|_2$ is equivalent to $\|\nabla\widetilde{u}(\cdot)\|_2$ (by the Calderon-Zygmund Theorem), we can simply invoke local-in-time space-time analytic smoothing of the NSE given in [FT].

*Proof.* Consider the enstrophy equation (4)

$$\frac{1}{2}\frac{d}{dt}\int |\omega|^2\,dx + \nu\int |\nabla\omega|^2\,dx = \int S\omega\cdot\omega\,dx = \int \alpha|\omega|^2\,dx = I$$

and write $I = I_{r^*} + I_{r^*,R^*} + I_{(R^*)^C}$. The first term is bounded using Proposition 5 and the last term is bounded using Proposition 6 in conjunction with the fact that $\|\omega\|_1$ is estimated in terms of the initial data uniformly in time [C1]. The $x$-integral in the middle term is then split in an integral over $\Omega_t(M^*)$ and an integral over $(\Omega_t(M^*))^C$. The first integral is 0 (utilizing i) and ii)) and the second is bounded in the same way as $I$ over $(\Omega_t(M^*))^C$ (using the fact that $M^*(\log M^*)^3 = \|\omega\|^{1/2} \in \mathrm{L}^1$, see [GR]). Inserting these bounds in the enstrophy equation yields

$$\frac{d}{dt}\int |\omega|^2\,dx \le c_{0,\nu}M^*(\log M^*)^3\int |\omega|^2\,dx$$

on $(0,T)$. Hence $\lim_{t\uparrow T}\|\omega(t)\|_2 < \infty$ and we can extend $\omega$ to $(0,T]$ – denote the extension by $\widetilde{\omega}$. Since $\|\widetilde{\omega}(T)\|_2 < \infty$, local-in-time well-posedness of (1) in $L^2$ yields the (smooth) extension beyond $T$. □

If we could handle the sub-level sets $(\Omega(M^*))^C$ for $M^*$ larger than in Theorem 7, we could take smaller $R^*$. That follows from the following simple estimate.

**Proposition 9.** *Let $x$ be restricted to $\Omega(\frac{1}{c}\|\omega\|_\infty^\delta)$ for some $\delta$ in $(0,1]$ and some constant $c > 1$. Then for $R \ge \|\omega\|_\infty^{-\delta/3}$,*

$$|I_{R^C}| \le c\|\omega\|_2^2\int |\omega|^2\,dx.$$

*Proof.*

$$\begin{aligned}|I_{R^C}| &\le \int |\omega(x)|^2\int_{|y|\ge R}\frac{|\omega(x+y)|^2}{R^3|\omega(x+y)|}\,dy\,dx\\ &\le c\int |\omega(x)|^2\int_{|y|\ge R}|\omega(x+y)|^2\,dy\,dx\\ &\le c\|\omega\|_2^2\int |\omega|^2\,dx.\end{aligned}$$

□

In particular, Proposition 9 implies that if $\Omega(M^*) \subseteq \Omega(\frac{1}{c}\|\omega\|_\infty)$, we can take $R^* = \|\omega\|_\infty^{-1/3}$ in Theorem 7.

The following theorem will be proved in the last section.

**Theorem 10.** *Let $\omega$ be a smooth solution of (1) on $(0,T)$ for some $T > 0$ corresponding to the initial data satisfying $u_0 \in L^2$ and $\omega_0 \in L^1 \cap L^2$. Let $r^*(t) \sim \frac{\nu^{1/2}}{\|\omega(t)\|_\infty^{1/2}}$, $M_1^*(t) \sim \|\omega(t)\|_2$ and $M_2^*(t) \sim \|\operatorname{curl}\omega(t)\|_{4/(3+\epsilon)}^{4/(3+\epsilon)}$ for some $0 < \epsilon \le \frac{1}{2}$, and assume that for every $t$ in $(0,T)$ a region of high fluid activity $\Omega_t(M_1^*(t)) \cap \Psi_t(M_2^*(t))$ consists of small-scale structures $\{W_\beta(t)\}$ ($\beta$ in some index set) satisfying the following two properties:*

*i)* $\operatorname{diam} W_\beta(t) \le \frac{1}{c} r^*(t)$,
*ii)* $\operatorname{dist}(W_\beta(t), W_\gamma(t)) \ge c,\ \beta \neq \gamma$,

*for an appropriate constant* $c > 2$. *Then* $\lim_{t \uparrow T} \|\omega(t)\|_2 < \infty$, *i.e.* $T$ *is not a singular time, and thus* $\omega$ *can be extended smoothly beyond* $T$.

*Remark 11.* A sparseness condition here is imposed only in a region where *both* $\omega$ and $\operatorname{curl}\omega$ are large.

## 4. Isotropy

A special case of *anisotropy*, i.e. 1/2-Hölder-like regularity of the vorticity direction suffices to prevent finite-time blow-up. It will be shown in this section that a certain *isotropy* condition controls the evolution of enstrophy.

We start by introducing another piece of notation. For $0 < r < \infty$ and $0 < r_1 < r_2 < \infty$ define $J_r$, $J_{r^C}$, $J_{r_1,r_2}$ and $S_r$ by

$$J_r = \int\int_{|y|\le r} (\omega(x)\cdot\hat{y})(\operatorname{curl}\omega(x+y)\cdot\omega(x))\frac{1}{|y|^2}\,dy\,dx,$$

$$J_{r^C} = \int\int_{|y|\ge r} (\omega(x)\cdot\hat{y})(\operatorname{curl}\omega(x+y)\cdot\omega(x))\frac{1}{|y|^2}\,dy\,dx,$$

$$J_{r_1,r_2} = \int\int_{r_1\le|y|\le r_2} (\omega(x)\cdot\hat{y})(\operatorname{curl}\omega(x+y)\cdot\omega(x))\frac{1}{|y|^2}\,dy\,dx,$$

and

$$S_r = \int\int_{|y|=r} (\omega(x)\cdot\hat{y})(\omega(x+y)\times\omega(x)\cdot\hat{y})\frac{1}{|y|^2}\,dS_y\,dx.$$

**Lemma 12.** *Let* $0 < r_1 < r_2 < \infty$. *Then*

$$I_{r_1,r_2} = \frac{1}{3}J_{r_1,r_2} - \frac{1}{3}S_{r_2} + \frac{1}{3}S_{r_1}.$$

*Proof.* Computing the $y$-surface integrals via the Divergence Theorem gives

$$\begin{aligned}
&\left(\int_{|y|=r_2} - \int_{|y|=r_1}\right)(\omega(x)\cdot\hat{y})(\omega(x+y)\times\omega(x)\cdot\hat{y})\frac{1}{|y|^2}\,dS_y\\
&= \left(\int_{|y|=r_2} - \int_{|y|=r_1}\right)\left(\frac{\omega(x)\cdot y}{|y|^3}\omega(x+y)\times\omega(x)\right)\cdot\hat{y}\,dS_y\\
&= \int_{r_1\le|y|\le r_2}\left(\frac{\omega(x)\cdot y}{|y|^3}\operatorname{div}(\omega(x+y)\times\omega(x))\right.\\
&\quad\left.+(\omega(x+y)\times\omega(x))\cdot\nabla\frac{\omega(x)\cdot y}{|y|^3}\right)dy\\
&= \int_{r_1\le|y|\le r_2}\left(\frac{\omega(x)\cdot\hat{y}}{|y|^2}(\operatorname{curl}\omega(x+y)\cdot\omega(x)) + \frac{\omega(x+y)\times\omega(x)\cdot\omega(x)}{|y|^3}\right.\\
&\quad\left.-3\frac{(\omega(x)\cdot\hat{y})(\omega(x+y)\times\omega(x)\cdot\hat{y})}{|y|^3}\right)dy.
\end{aligned}$$

Noting that the second term in the last integral is zero, and integrating in $x$ yields the desired identity. □

Control of the surface terms on integral and small scales is given in the following two lemmas.

**Lemma 13.** *Let* $0 < \epsilon \le \frac{1}{2}$. *Then*

$$|S_1| \le c \left(1 + \|\omega\|_1 + \|\operatorname{curl}\omega\|_{4/(3+\epsilon)}^{4/(3+\epsilon)}\right) \int |\omega|^2 \, dx.$$

*Proof.* The following manipulations are elementary.

$$\begin{aligned}
&\int \int_{|y|=1} (\omega(x)\cdot\hat{y})(\omega(x+y)\times\omega(x)\cdot\hat{y})\frac{1}{|y|^2}\, dS_y\, dx \\
&= -\int \omega(x)\cdot \int_{|y|=1} ((\omega(x)\cdot y)\omega(x+y))\times\hat{y})\, dS_y\, dx \\
&= \int \omega(x)\cdot \int_{|y|\le 1} \operatorname{curl}\ ((\omega(x)\cdot y)\omega(x+y))\ dy\, dx \\
&= \int \omega(x)\cdot \int_{|y|\le 1} ((\omega(x)\cdot y)\operatorname{curl}\omega(x+y) - \omega(x+y)\times\nabla(\omega(x)\cdot y))\ dy\, dx \\
&= \int \omega(x)\cdot \int_{|y|\le 1} ((\omega(x)\cdot y)\operatorname{curl}\omega(x+y) - \omega(x+y)\times\omega(x))\ dy\, dx \\
&\le \int |\omega(x)|^2 \int_{|y|\le 1} |\operatorname{curl}\omega(x+y)|\, dy\, dx + \int |\omega(x)|^2 \int_{|y|\le 1} |\omega(x+y)|\, dy\, dx.
\end{aligned}$$

□

**Lemma 14.** *Let* $\nu > 0$ *and* $r \le \frac{r^*}{2}$. *Then*

$$|S_r| \le \frac{\nu}{4}\int |\nabla\omega|^2\, dx.$$

*Proof.* As in the proof of Proposition 5 we can show that

$$\begin{aligned}
S_r &= \int\int_{|y|=r} (\omega(x)\cdot\hat{y})(\omega(x+y)\times\omega(x)\cdot\hat{y})\frac{1}{|y|^2}\, dS_y\, dx \\
&= \frac{1}{2}\int\int_{|y|=r} ((\omega(x)-\omega(x+y))\cdot\hat{y})(\omega(x+y) \\
&\quad \times(\omega(x)-\omega(x+y))\cdot\hat{y})\frac{1}{|y|^2}\, dS_y\, dx, \qquad (8)
\end{aligned}$$

and thus

$$\begin{aligned}
|S_r| &= \left|\int\int_0^1 ds_1 \int_0^1 ds_2 \int_{|y|=r} (\nabla\omega(x+s_1 y)\hat{y}\cdot\hat{y})(\omega(x+y)\times\nabla\omega(x+s_2 y)\hat{y}\cdot\hat{y}) dS_y dx\right| \\
&\le \int\int_0^1 ds_1 \int_{|y|=r} |\nabla\omega(x+s_1 y)|^2 \|\omega\|_\infty dS_y dx \le 4\pi r^2 \|\omega\|_\infty \int |\nabla\omega(z)|^2 dz. \qquad (9)
\end{aligned}$$

□

For $x$ in $\Omega(M^*)$ consider an orthonormal triplet $\{e_1, e_2, e_3\}$, where $e_1 = \xi(x)$, and define the fluxes of the orthogonal projections of $\operatorname{curl}\omega$ by

$$I_r^i(x) = \int_{|y|=r} [\operatorname{curl}\omega(x+y)\cdot e_i)e_i]\cdot \hat{y}\, dS_y$$

for $r > 0$, $i = 1, 2, 3$.

Then we have the following theorem.

**Theorem 15.** *Let $\omega$ be a smooth solution of (1) on $(0, T)$ for some $T > 0$ corresponding to the initial data satisfying $u_0 \in L^2$ and $\omega_0 \in L^1 \cap L^2$. Assume that for every $t$ in $(0, T)$, every $x$ in $\Omega_t(M^*(t))$, and all $r^*(t) \le r(t) \le 1$,*

$$|I^i_{r(t)}(x) - I^j_{r(t)}(x)| \le c$$

*for some absolute constant $c$. Then $\lim_{t\uparrow T} \|\omega(t)\|_2 < \infty$, and thus $\omega$ can be extended smoothly beyond $T$.*

*Proof.* Similarly as in the proof of Theorem 7 decompose $I$ as $I = I_{r^*} + I_{r^*,1} + I_{1^C}$. Hence only the restriction of the middle term to $\Omega_t(M^*)$ needs to be estimated. Observing that Lemma 12 holds on any $x$-subset of $\mathbb{R}^3$, write $I^*_{r^*,1} = \frac{1}{3}J^*_{r^*,1} - \frac{1}{3}S^*_1 + \frac{1}{3}S^*_{r^*}$, where a superscript $*$ denotes a restriction of the corresponding integral to $\Omega_t(M^*)$. Since $S^*_1 \le S_1$, $S^*_{r^*} \le S_{r^*}$ and they can be bounded in the same way as $S_1$ and $S_{r^*}$ (cf. Lemmas 13 and 14), we are left with $J^*_{r^*,1}$ which can be written as

$$\int_{\Omega_t(M^*)} |\omega(x)|^2 \int_{r^*}^1 \frac{1}{\rho^2} \int_{|y|=\rho} [(\operatorname{curl}\omega(x+y)\cdot\xi(x))\xi(x)]\cdot\hat{y}\, dS_y\, d\rho\, dx. \qquad (10)$$

Notice that by the choice of $e_1$ the surface integral is equal to $I^1$.

We claim that $I^1 + I^2 + I^3 = 0$. Observe that

$$\int_{|y|=\rho} \operatorname{curl}\omega(x+y)\cdot\hat{y}\, dS_y = \int_{|y|\le\rho} \operatorname{div}\operatorname{curl}\omega(x+y)\, dy = 0.$$

Expanding $\hat{y}$ as $\hat{y} = (e_1\cdot\hat{y})e_1 + (e_2\cdot\hat{y})e_2 + (e_3\cdot\hat{y})e_3$ yields the claim.

Combining the cancellation relation $I^1 + I^2 + I^3 = 0$ with the isotropy assumption $|I^i - I^j| \le c$ gives $|I^1| \le c$. Inserting this in (10) finishes the proof. □

*Remark 16.* It is worth noticing that $\operatorname{curl}\omega = -\Delta u$, and since the Laplacian is a rotationally invariant operator, the condition in the theorem can be viewed as an isotropy condition on the velocity field $u$.

## 5. Proof of Theorem 10

*Proof.* Consider the enstrophy equation (4)

$$\frac{1}{2}\frac{d}{dt}\int |\omega|^2\, dx + \nu\int |\nabla\omega|^2\, dx = \int S\omega\cdot\omega\, dx = \int \alpha|\omega|^2\, dx = I$$

and write $|I| \le |I_{r^*}| + |S_{r^*}| + |J_{r^*,1}| + |S_1| + |I_{1^C}|$. As in the previous proofs, only $J_{r^*,1}$ remains to be estimated.

To this effect, decompose both $\omega$ and $\operatorname{curl}\omega$ in the low and high magnitude parts. Let $\Xi_A$ be a smooth cut-off function over a set $A$. We write $\omega(t) = \omega_1(t) + \omega_2(t)$, where

$$\omega_1(t) = (1 - \Xi_{\Omega_t(M_1^*(t))})\omega(t) \text{ and } \omega_2(t) = \Xi_{\Omega_t(M_1^*(t))}\omega(t),$$

and similarly, $\operatorname{curl}\omega(t) = (\operatorname{curl}\omega)_1(t) + (\operatorname{curl}\omega)_2(t)$, where

$$(\operatorname{curl}\omega)_1(t) = (1 - \Xi_{\Psi_t(M_2^*(t))}) \operatorname{curl}\omega(t) \text{ and } (\operatorname{curl}\omega)_2(t) = \Xi_{\Psi_t(M_2^*(t))} \operatorname{curl}\omega(t).$$

Then

$$\begin{aligned}
|J_{r^*,1}| &\le \int |\omega(x)|^2 \int_{r^*}^1 \frac{|\operatorname{curl}\omega(x+y)|}{|y|^2}\,dy\,dx \\
&\le 2\Big( \int |\omega_1(x)|^2 \int_{r^*}^1 \frac{|(\operatorname{curl}\omega)_1(x+y)|}{|y|^2}\,dy\,dx \\
&\quad + \int |\omega_1(x)|^2 \int_{r^*}^1 \frac{|(\operatorname{curl}\omega)_2(x+y)|}{|y|^2}\,dy\,dx \\
&\quad + \int |\omega_2(x)|^2 \int_{r^*}^1 \frac{|(\operatorname{curl}\omega)_1(x+y)|}{|y|^2}\,dy\,dx \\
&\quad + \int |\omega_2(x)|^2 \int_{r^*}^1 \frac{|(\operatorname{curl}\omega)_2(x+y)|}{|y|^2}\,dy\,dx \Big) \\
&= 2\left(J^{1,1} + J^{1,2} + J^{2,1} + J^{2,2}\right).
\end{aligned}$$

For $J^{1,1}$ and $J^{1,2}$ the integral is bounded (via Hölder and Hardy-Littlewood-Sobolev inequalities) by

$$\begin{aligned}
&(M_1^*)^{1/3} \int |\omega(x)|^{5/3} \int \frac{|\operatorname{curl}\omega(x+y)|}{|y|^2}\,dy\,dx \\
&\le c(M_1^*)^{1/3} \left( \int |\omega|^2\,dx \right)^{5/6} \|\operatorname{curl}\omega\|_2 \\
&\le \frac{c}{\nu}(M_1^*)^{2/3} \left( \int |\omega|^2\,dx \right)^{5/3} + \frac{\nu}{2} \int |\nabla\omega|^2\,dx \\
&\le \frac{c}{\nu}\|\omega\|_2^2 \int |\omega|^2\,dx + \frac{\nu}{4} \int |\nabla\omega|^2\,dx.
\end{aligned}$$

For $J^{2,1}$ the integral is bounded by

$$M_2^*(1-r^*) \int |\omega|^2\,dx \le c\|\operatorname{curl}\omega\|_{4/(3+\epsilon)}^{4/(3+\epsilon)} \int |\omega|^2\,dx.$$

Finally, for $J^{2,2}$, we are in a region of high fluid activity and according to our geometric assumptions i) and ii) the integral is equal to 0.

Collecting all the estimates,

$$I \le c\left(1 + \|\omega\|_1 + \frac{1}{\nu}\|\omega\|_2^2 + \|\operatorname{curl}\omega\|_{4/(3+\epsilon)}^{4/(3+\epsilon)}\right) \int |\omega|^2\,dx + \nu \int |\nabla\omega|^2\,dx.$$

Since the expression in the parantheses is in $L^1(0,T)$, inserting this inequality in the enstrophy equation finishes the proof. □

*Remark 17.* For simplicity of the exposition the cut-off for the large scales in the proofs was taken to be $O(1)$. The same line of reasoning applies for a cut-off of the order of $R(t)$, where $R(t)$ satisfies

$$\frac{1}{R(\cdot)^{\frac{9+3\epsilon}{1-\epsilon}}} \in L^1(0, T)$$

for some $0 < \epsilon \leq \frac{1}{2}$.

*Acknowledgements.* We thank the referee for a number of comments and suggestions that led to the present version of the paper.

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Communicated by P. Constantin