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Existence of the Energy-Level Weak Solutions for a Nonlinear Fluid-Structure Interaction Model

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ABSTRACT. A 3D fluid-structure interaction model in which an elastic body is fully immersed in a viscous incompressible fluid is studied. The interaction is realized through an interface, i.e., the boundary of the elastic body. More precisely, continuity of the velocities and the normal components of the stress tensors across the interface is required. The main result is global-in-time existence of weak solutions in the natural energy-level class.

1. Introduction

We consider a model of fluid-structure interaction on a bounded domain $\Omega \in R^n$, $n = 2, 3$, where Ω is comprised of two open domains Ω_f and Ω_s. A stationary elastic solid Ω_s is fully immersed in a fluid occupying domain Ω_f with interaction taking place on the boundary of the solid Γ_s. The dynamics of the solid is described by a linear elastic equation (hyperbolic) in the variable w, while the dynamics of the fluid are described by the Navier-Stokes equations (NSE) (parabolic) in the variables u and p (the velocity of the fluid and the pressure, respectively). The interaction between the two systems takes place on the boundary Γ_s that is common to both media and is realized via suitable (Neumann type) transmission boundary conditions. Continuity of both the velocities and the normal components of the stress tensors across Γ_s is required.

The model presented is well established in both physical and mathematical literature [**24, 15, 10, 13, 14**]. From the physical point of view, it is an important model arising in a variety of applications such as cell biology, mechanics or fluid dynamics. From the mathematical point of view, the interest of the model stems from rather unusual functional analytic setup that is not amenable to standard variational analysis–as in the case of the NSE or the wave equations *per se*. This is due to boundary conditions which are of Neumann type and, as such, can not be treated via standard Leray projection to divergence-free spaces. The presence of the pressure, both in the equation and on the boundary is a characteristic feature of the problem. This, in turn, leads to weak formulation that involves traces of a wave equation that are not classically defined in the topology of the natural energy-level weak solutions. Classical mismatch of regularity between parabolic and hyperbolic

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equations/systems **[10]** is at the heart of the problem. It is worth mentioning that this particular issue does not arise in the context of fluid-rigid body interactions, where the wave equation is replaced by a suitable nonlinear ordinary differential equation **[25, 14, 12]**. Mathematical analysis of the resulting coupled system is very different.

In order to deal with the issue of "mismatch " of parabolic-hyperbolic regularity, some results of the previous literature have considered an "approximation" of wave dynamics where an additional *smoothing effect*, in terms of a structural damping, is added to the dynamics of the wave equation, c.f. **[7, 8]** and references therein. For such models, the wave equation displays enough regularity for standard trace theory to be applicable. On the other end of the spectrum, recent developments in the field for the unperturbed model were focused exclusively on *very smooth* local-in-time solutions which lead to topologies where, again, the classical traces are well defined **[10, 14]**.

In view of the above, a distinct feature of this paper is that we consider *weak solutions* of the structure *without any regularizing effects* for the wave dynamics. For this problem, we establish existence of global weak solutions in the natural *energy-level* class. In particular, our results recover optimal results established for the NSE *per se*, but in the context of the fluid-structure interaction with transmission coupling on the boundary.

As to the mathematical difficulties and challenges, one of the main obstacles is a good understanding of trace theory corresponding to weak solutions of the interaction, and the other one is establishing existence for the structure component without any reliance on Leray projector. The first one necessitates a careful micro-local analysis treatment of the wave component. Only then, one can give a rigorous meaning to weak solutions corresponding to the interaction–this is due to the presence of the normal component of the elastic stress tensor on the interface Γ_s. The second difficulty is handled by a very particular use of monotone operator theory **[4]** (the problem is not monotone) followed by appropriate approximations and limit passages. This last step also relies on the special regularity of the strain tensor on the interface.

The model under consideration is the following. Let $\Omega \in R^n, n = 2, 3$, be a bounded domain with an interior region Ω_s (a domain occupied by an elastic solid) and an exterior region Ω_f (a domain filled with viscous incompressible fluid). Denote by Γ_f the outer boundary of the domain Ω and by Γ_s the boundary of the region Ω_s which is also an interior boundary of Ω_f, and where the interaction of the two systems takes place. Let u be a vector function defined on $\Omega_f \times [0, T]$ representing the velocity of the fluid and p a scalar function representing the pressure. Additionally, let w, w_t be the displacement and velocity functions of the elastic solid Ω_s. We also denote by ν the unit outward normal vector on Γ_s with respect to the region Ω_s.

We are seeking a quadruple (u, w, w_t, p) satisfying the following system:

$$(1.1)\qquad \begin{cases} u_t - \operatorname{div} \epsilon(u) + (u \cdot \nabla)u + \nabla p = 0 & \Omega_f \times (0,T) \\ \operatorname{div}\, u = 0 & \Omega_f \times (0,T) \\ w_{tt} - \operatorname{div}\, \sigma(w) = 0 & \Omega_s \times (0,T) \\ u(0,\cdot) = u_0 & \Omega_f \\ w(0,\cdot) = w_0,\ w_t(0,\cdot) = w_1 & \Omega_s \\ u = 0 & \Gamma_f \times (0,T) \\ w_t = u & \Gamma_s \times (0,T) \\ \sigma(w)\cdot \nu = \epsilon(u).\nu - p\nu - \frac{1}{2}(u\cdot\nu)u & \Gamma_s \times [0,T] \end{cases}$$

where the elastic stress tensor σ and the strain tensor, respectively, are given by

$$\sigma_{ij}(u) = \lambda \sum_{k=1}^{k=3} \epsilon_{kk}(u)\delta_{ij} + 2\mu\epsilon_{ij}(u), \lambda, \mu > 0, \quad \text{and} \quad \epsilon_{ij}(u) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}).$$

In the system considered, the interface Γ_s is stationary. This corresponds to a physical situation in which the order of magnitude of the displacement of the elastic solid on the boundary is smaller than the order of magnitude of the velocity (small but rapid oscillations). From the mathematical point of view, if one considers the case of a moving interface, the equation for the interface in the Lagrangian coordinates is comparable to our elastic equation **[10]**, and the core of the problem arising from the parabolic-hyperbolic coupling across the interface via the continuity of the velocities and the normal components of the stress tensors remains essentially the same.

Also, note that the presence of the (not necessarily small) fluid term $\frac{1}{2}(u\cdot\nu)u$ on Γ_s is due to the fact that the interface is stationary. Namely, in addition to the normal component of the usual fluid stress tensor $T = -pI + \epsilon(u)$ where ϵ is the deformation tensor, i.e., the symmetric part of the gradient, which would be present in the case of the moving interface as well, this model features an additional stress exerted on the interface originating in the tendency of the fluid to advect through the interface. The advection (transport) term in the Navier-Stokes equations is $(u\cdot\nabla)u$ which is, due to the incompressibility of the fluid, equal to div $(u \otimes u)$, and that exactly corresponds to a boundary term of the $(u\otimes u)\nu = (u\cdot\nu)u$-type. Requiring that the total energy of the system be non-increasing implies that the correct scaling factor for this boundary term is $1/2$ (see also **[21]**). Naturally, this is essential in deriving *a priori* bounds on the energy-level weak solutions. In the case of non-stationary interface, this boundary term is entirely absorbed by the material derivative of the velocity of the fluid, which accounts for the dynamics of the spatial coordinate. This is the reason why boundary conditions in the stationary and non-stationary case differ by precisely the quadratic boundary term in question. In conclusion: the presence of this quadratic boundary term is essential in the model where the interface undergoes (small but rapid) oscillations in, however, geometrically fixed position. This is due to the requirement that the total energy does not increase.

It is our hope that the analysis of the present paper will allow to obtain analogous results for the model with the solid *moving* in the fluid (see**[24]**).

Throughout the paper $\mathcal{H} \equiv H \times H^1(\Omega_s) \times L_2(\Omega_s)$ where

$$H \equiv \{u \in L_2(\Omega_f) : \operatorname{div}\, u = 0, u\cdot\nu|_{\Gamma_f} = 0\}$$

will denote the energy space for the system.

Note that all Sobolev spaces H^s, L_2 pertaining to u and w are in fact $(H^s)^n, (L_2)^n, n = 2, 3$ and only for simplicity we omit the exponent n.

In addition we will use the following notation:

$$V \equiv \{v \in H^1(\Omega_f) : \operatorname{div} v = 0, u|_{\Gamma_f} = 0\}$$

$$(u, v) = \int_\Omega uv \, d\Omega, \ \langle u, v\rangle = \int_{\Gamma_s} uv \, d\Gamma_s, \ D_i = \frac{\partial}{\partial x_i}$$

$$|u|_{s,D} = |u|_{H^s(D)}; \ |u|_s = |u|_{s,\Omega}, \ |u| = |u|_{0,\Omega}.$$

V is topologized with respect to the inner product given by:

$$(u, v)_{1,f} \equiv \int_{\Omega_f} \epsilon(u)\epsilon(v) d\Omega_f$$

We denote the induced norm by $|.|_{1,\Omega_f}$ which is a norm equivalent to the usual $H^1(\Omega_f)$ norm via Korn's inequality and Poincare's inequality

$$|u|_{1,\Omega_f} = [\int_{\Omega_f} |\epsilon(u)|^2 d\Omega_f]^{1/2} \quad .$$

$H^1(\Omega_s)$ is topologized with respect to the inner product given by

$$(w, z)_{1,s} \equiv \int_{\Omega_s} wz + \int_{\Omega_s} \sigma(w)\epsilon(z)$$

We denote by $|.|_{1,\Omega_s}$ the induced norm by inner product above

$$|w|^2_{1,\Omega_s} = [\int_{\Omega_s} \sigma(w)\epsilon(w) d\Omega_s] + |w|^2_{0,\Omega_s}$$

The norm $|.|_{1,\Omega_s}$ is again equivalent to the usual $H^1(\Omega_s)$ norm by Korn's inequality.

Finally, the Energy functional for the system is

$$E(t) \equiv |u(t)|^2_{0,\Omega_f} + (\sigma(w(t)), \epsilon(w(t)))_{\Omega_s} + |w_t(t)|^2_{0,\Omega_s}$$

2. The Main Result

DEFINITION 2.1. Let $[u_0, w_0, w_1] \in \mathcal{H}$ and $T > 0$. We say that a triple $[u, w, w_t] \in L_\infty\left((0,T); H \times H^1(\Omega_s) \times L_2(\Omega_s)\right) = L_\infty((0,T); \mathcal{H})$ is a weak solution of (1.1) if

- $(u(0), w(0), w_t(0)) = (u_0, w_0, w_1)$ (in the sense of weak continuity),
- $\sigma(w).\nu \in L_2((0,T); H^{-1/2}(\Gamma_s))$,
- $w_t|_{\Gamma_s} = u|_{\Gamma_s} \in L_2((0,T), H^{1/2}(\Gamma_s))$
- and the following variational system holds a.e in $t \in (0,T)$

$$(u_t, \phi) + (\epsilon(u), \epsilon(\phi)) + ((u.\nabla)u, \phi) + \langle\sigma(w).\nu, \phi\rangle - \tfrac{1}{2}\langle(u.\nu)u, \phi\rangle = 0 \tag{2.1}$$
$$(w_{tt}, \psi) + (\sigma(w), \epsilon(\psi)) - \langle\sigma(w).\nu, \psi\rangle = 0$$

for all test functions $\phi \in V$ and $\psi \in H^1(\Omega_s)$.

The above definition of weak solution is very natural. In fact, by projecting the original equations in (1.1) on H and using boundary conditions, the variational form in (2.1) is precisely the result of that projection.

THEOREM 2.2. *Given any initial condition* $[u_0, w_0, w_1] \in \mathcal{H}$ *and any* $T > 0$*, there exists a weak solution* $[u, w, w_t]$ *to the system (1.1), i.e.* $[u, w, w_t] \in L_\infty((0,T); \mathcal{H})$ *such that*

$$\begin{aligned} u &\in L_2((0,T); V) \\ \nabla w|_{\Gamma_s} &\in L_2((0,T); H^{-1/2}(\Gamma_s)) \\ w_t|_{\Gamma_s} &\in L_2((0,T); H^{1/2}(\Gamma_s)) \end{aligned} \tag{2.2}$$

Moreover, in the case when dimension of $\Omega = 2$*, weak solutions are unique within the class specified above.*

The remainder of this paper is devoted to the proof of Theorem 2.2. For reader's orientation, we shall outline below the main steps of the proof.

In order to prove existence of weak solutions we shall use a combination of nonlinear semigroup methods along with variational and weak compactness methods. The main steps in the proof are as follows.

Step 1. A first step is to consider an "auxiliary" problem which can be interpreted as a very special approximation of the original problem. This approximation is ω m-accretive. Nonlinear semigroup theory will yield global solvability for this auxiliary problem. The solution furnished by the semigroup theory is the so called"semigroup" or "mild" solution.

Step 2. The crux of the proof is to show that this semigroup solution satisfies a suitable variational form. It is at this point where the regularity of hyperbolic traces becomes critical. Utilizing micro-local analysis methods we are able to secure $L_2((0,T); H^{-1/2}(\Gamma_s))$ regularity of transmission conditions.

Step 3. Having obtained a correct variational formulation of the semigroup solutions corresponding to the auxiliary problem, we construct the appropriate auxiliary problem for our system by a suitable truncation of the nonlinear term in the NSE. Passage to the limit using weak compactness methods leads to variational formulation of the full nonlinear problem.

REMARK 2.1. It may be interesting to note that in *the linear case*, the well-posedness of finite energy solutions does not require "good definition" of the strain tensor on the interface. Indeed, as shown in [**3**], semigroup based approach leads to existence of a C_0 semigroup solution without necessity for characterization of transmission conditions. However, if one is to obtain variational characterization of the said solution (essential for the treatment of nonlinear problem) , additional information and analysis of transmission conditions is necessary.

REMARK 2.2. An interesting problem associated with the problem studied in this paper is stability analysis (when $t \to \infty$) of finite energy solutions. In the *linear* case, this has been carried out in a recent manuscript [**3, 2**].

3. The proof of Theorem 2.2

3.1. An Auxiliary Problem. Consider the following auxiliary problem, referred as *Problem-L*:

$$(u_t, \phi) + (\epsilon(u), \epsilon(\phi)) + (Lu, \phi) + \langle \sigma(w) \cdot \nu, \phi \rangle = 0 \tag{3.1}$$

$$w_t|_{\Gamma_s} = u|_{\Gamma_s}, \ in \ L_2((0,T), H^{1/2}(\Gamma_s)) \tag{3.2}$$

$$(w_{tt}, \psi) + (\sigma(w), \epsilon(\psi)) - \langle \sigma(w) \cdot \nu, \psi \rangle = 0 \tag{3.3}$$

$\forall \phi \in V, \ \psi \in H^1(\Omega_s) \ a.e. \ in \ t \in (0,T)$.

The nonlinear operator $L : V \to V'$ is assumed to have the following properties:

- $L : V \to V'$ is locally Lipschitz and $L(0) = 0$
- $\forall \epsilon > 0, \exists C_\epsilon$ such that

$$|(Lu - Lv, u - v)| \leq \epsilon |u - v|^2_{1,\Omega_f} + C_\epsilon |u - v|^2_{0,\Omega_f}, \forall u, v \in V \tag{3.4}$$

We introduce the operator $A_L : V \times H^{-1/2}(\Gamma_s) \to V'$ defined by

$$(A_L(u,z), \phi) \equiv (\epsilon(u), \epsilon(\phi)) + \langle z \cdot \nu, \phi \rangle + (Lu, \phi)_{V',V} \tag{3.5}$$

for all $\phi \in V$.

We can rewrite (3.1), (3.3) in the new notation as

$$(u_t, \phi) + (A_L(u, \sigma(w)), \phi) = 0$$

$$(w_{tt}, \Psi) - (\operatorname{div} \sigma(w), \Psi) = 0$$

$$w_t|_{\Gamma_s} = u|_{\Gamma_s}$$

The next step is to represent *Problem-L* as an abstract nonlinear evolution. For this we set: $Y \equiv (u, w, z) \in \mathcal{H}$

$$Y_t = \mathcal{A}_L Y, \ Y_0 \in \mathcal{H}$$

with

$$\begin{pmatrix} u \\ w \\ z \end{pmatrix}_t = \mathcal{A}_L Y = \begin{pmatrix} -A_L(u, \sigma(w)) \\ z \\ \operatorname{div} \sigma(w) \end{pmatrix} \tag{3.6}$$

where $\mathcal{D}(\mathcal{A}_L)$ is given by

$$\begin{aligned} \mathcal{D}(\mathcal{A}_L) = \{(u,w,z) \in \mathcal{H} : u \in V, A_L(u, \sigma(w)) \in H, z \in H^1(\Omega_s), \\ \operatorname{div} \sigma(w) \in L_2(\Omega_s), z|_{\Gamma_s} = u|_{\Gamma_s} \ in \ H^{1/2}(\Gamma_s)\} \end{aligned} \tag{3.7}$$

REMARK 3.1. We note that for elements $(u, w, z) \in \mathcal{D}(\mathcal{A}_L)$ we also have that $\sigma(w) \cdot \nu \in H^{-1/2}(\Gamma_s)$. Indeed, this follows from $div\sigma(w) \in L_2(\Omega_s)$ along with a priori regularity of $w \in H^1(\Omega_s)$. Green's formula and surjectivity properties of trace operators allow to infer the needed trace regularity available for solutions to elliptic problems.

We shall prove that the operator $\mathcal{A}_L$ generates a c_0 semigroup of contractions on $\mathcal{H}$.

PROPOSITION 3.1. For each operator L satisfying properties (3.4) , the operator $\mathcal{A}_L$ with $\mathcal{D}(\mathcal{A}_L)$ given in (3.7) generates a strongly continuous nonlinear semigroup on $\mathcal{H}$.

PROOF. The result follows once we show that $-\mathcal{A}_L$ **is** ω **m-accretive.**
Step 1: *ω-accretivity*: with $u = u_1 - u_2$, $w = w_1 - w_2$, $z = z_1 - z_2 \in \mathcal{D}(\mathcal{A}_L)$:

$$\langle -\mathcal{A}_L \begin{pmatrix} u_1 \\ w_1 \\ z_1 \end{pmatrix} + \mathcal{A}_L \begin{pmatrix} u_2 \\ w_2 \\ z_2 \end{pmatrix}, \begin{pmatrix} u \\ w \\ z \end{pmatrix} \rangle_{\mathcal{H}} =$$

$$\begin{aligned}(\epsilon(u), \epsilon(u)) + \langle \sigma(w).\nu, u\rangle - (\sigma w, \epsilon z) - \langle \sigma(w).\nu, u\rangle + (\sigma w, \epsilon z) + (Lu_1 - Lu_2, u)\\ = |\epsilon(u)|^2_{0,\Omega_f} + (Lu_1 - Lu_2, u_1 - u_2)\\ \geq |u|^2_{1,\Omega_f} - \epsilon|u_1 - u_2|^2_{1,\Omega_f} - C_\epsilon|u_1 - u_2|^2_{0,\Omega_f}\\ = |u|^2_{1,\Omega_f} - \epsilon|u|^2_{1,\Omega_f} - C_\epsilon|u|^2_{0,\Omega_f}\end{aligned}$$

Now, consider the full operator $\omega I - \mathcal{A}_L$ instead of $-\mathcal{A}_L$ to obtain :

$$\begin{aligned}(-\mathcal{A}_L(u_1, w_1, z_1) + \mathcal{A}_L(u_2, w_2, z_2), [u, w, z])_{\mathcal{H}} + \omega|u, w, z|^2_{\mathcal{H}}\\ \geq |u|^2_{1,\Omega_f} - \epsilon|u|^2_{1,\Omega_f} - C_\epsilon|u|^2_{0,\Omega_f} + \omega|u|^2_{0,\Omega_f}\end{aligned}$$

Choose $\epsilon > 0$ so that $1 - \epsilon > 0$, and choose ω so that $\omega - C_\epsilon > 0$. Thus we have:

$$(-\mathcal{A}_L(u_1, w_1, z_1) + \mathcal{A}_L(u_2, w_2, z_2), [u, w, z])_{\mathcal{H}} + \omega|u, w, z|^2_{\mathcal{H}} \geq 0$$

And ω-accretivity of $-\mathcal{A}_L$ follows.

Step 2: *Maximality.* We need to show that $\lambda Y - \mathcal{A}_L Y$ is surjective onto $\mathcal{H}$ for λ large enough. Let $\lambda Y - \mathcal{A}_L Y = [f, g, h] \in \mathcal{H}$, so we have:

$$\lambda u + A(u, \sigma(w)) + L(u) = f, \ \lambda w - z = g \tag{3.8}$$

$$\lambda z - \operatorname{div} \sigma(w) = h, \ z|_{\Gamma_s} = u|_{\Gamma_s} \tag{3.9}$$

or equivalently:

$$\lambda u + A(u, \sigma(w)) + L(u) = f, \ \lambda w - z = g \tag{3.10}$$

$$\lambda^2 w - \operatorname{div} \sigma(w) = \lambda g + h, \ w|_{\Gamma_s} = \frac{1}{\lambda}(u|_{\Gamma_s} + g|_{\Gamma_s}) \tag{3.11}$$

Now we can use elliptic theory to conclude that given $u \in V$, $g \in H^1(\Omega_s)$ and $h \in L_2(\Omega_s)$, there exists $w \in H^1(\Omega_s)$ such that w satisfies the elliptic system:

$$\lambda^2 w - \text{div } \sigma(w) = \lambda g + h \in H^{-1}(\Omega_s)$$

$$w|_{\Gamma_s} = \frac{1}{\lambda}(u|_{\Gamma_s} + g|_{\Gamma_s}) \in H^{1/2}(\Gamma_s)$$

As a result of elliptic theory, the solution to the above elliptic problem w and the normal trace on the boundary are continuously dependent on the data of the problem f, g, u with the following inequality:

$$|w|_{1,\Omega_s} + |\sigma(w).\nu|_{-1/2,\Gamma_s} \leq c_\lambda[|u|_{1/2,\Gamma_s} + |g|_{1/2,\Gamma_s} + |g|_{-1,\Omega_s} + |h|_{-1,\Omega_s}] \tag{3.12}$$

Therefore:

$$|w|_{1,\Omega_s} + |\sigma(w).\nu|_{-1/2,\Gamma_s} \leq c_\lambda[|u|_{1/2,\Gamma_s} + |g|_{1,\Omega_s} + |h|_{-1,\Omega_s}] \tag{3.13}$$

Notice that w can be decomposed into $w(u, g, h) = w(u, 0, 0) + w(0, g, h)$, so going back to the first equation in u, we can rewrite (3.8) as:

$$\lambda u + A(u, \sigma(w(u, 0, 0))) + A(0, \sigma(w(0, g, h))) + Lu = f$$

We will denote $w(0,g,h)$ by w_{gh} and $w(u,0,0)$ by w_u. Since the existence of $w(0,g,h)$ is guaranteed, we need to find $u \in V$ satisfying

$$\lambda u + A(u,\sigma(w_u)) + L(u) = -A(0,\sigma(w_{gh}))) + f. \tag{3.14}$$

In other words, we need to show that the operator $\lambda I + A + L : V \to V'$ is surjective for λ large enough. To that end, it suffices to show that it is continuous, monotone and coercive in u. [**4**]

Continuity of $\lambda I + A_L$.

$$\begin{aligned}\lambda(u,\phi) + (A(u,\sigma(w_u)),\phi) &= \lambda(u,\phi) + (\epsilon(u),\epsilon(\phi)) + \langle\sigma(w_u).\nu,\phi\rangle \\ &\le C[|u|_{1,\Omega_f}|\phi|_{1,\Omega_f} + |\sigma(w_u)|_{-1/2,\Gamma_s}|\phi|_{1/2,\Gamma_s}]\end{aligned}$$

But by estimate in (3.13) for $\sigma(w_u)$ and noting that $g,h=0$ in this case :

$$\begin{aligned}\lambda(u,\phi) + (A(u,\sigma(w_u))),\phi) &\le C[|u|_{1,\Omega_f}|\phi|_{1,\Omega_f} + c_\lambda|u|_{1/2,\Gamma_s}|\phi|_{1,\Omega_f}] \\ &\le C|u|_{1,\Omega_f}|\phi|_{1,\Omega_f}\end{aligned}$$

Thus, $\lambda I + A$ is continuous. Since $L : V \to V'$ is assumed Lipschitz, continuity of $\lambda I + A + L$ follows.

Coercivity of $\lambda I + A_L$.

$$\lambda(u,u) + (A(u,w_u),u) + (L(u),u) = \lambda|u|^2_{0,\Omega_f} + |\epsilon(u)|^2_{0,\Omega_f} + \langle\sigma(w_u).\nu,u\rangle + (L(u),u) \tag{3.15}$$

We can reexpress $\sigma(w_u)$ from (3.8- 15) with $g,h=0$ as follows:

$$\begin{gathered}\lambda w_u - z(u) = 0 \\ \lambda z(u) - \operatorname{div}\sigma(w_u) = 0 \\ z(u)|_{\Gamma_s} = u|_{\Gamma_s}\end{gathered}$$

Now, apply the stress tensor $\epsilon(.)$ to both sides of first equation, and take the inner product of the second with z:

$$\lambda\epsilon(w_u) - \epsilon(z(u)) = 0 \tag{3.16}$$

$$\lambda|z(u)|^2_{0,\Omega_s} - \langle\sigma(w_u).\nu, z(u)\rangle + (\sigma(w_u),\epsilon(z(u))) = 0. \tag{3.17}$$

Hence, substituting (3.16) into (3.17) we obtain:

$$\lambda|z(u)|^2_{0,\Omega_s} - \langle\sigma(w_u).\nu, z(u)\rangle + \lambda(\sigma(w_u),\epsilon(w_u)) = 0.$$

Thus for $\lambda \ge 0$

$$\langle\sigma(w_u).\nu, z(u)\rangle = \lambda|z(u)|^2_{0,\Omega_s} + \lambda(\sigma(w_u),\epsilon(w_u)) \ge 0 \tag{3.18}$$

where the above inequality follows from Korn's inequality. Therefore, using (3.18) and the property (3.1), (3.15) becomes

$$\lambda(u,u) + (A_L(u,w_u)) \ge (\lambda - C_\epsilon)|u|^2_{0,\Omega_f} + (1-\epsilon)|u|^2_{1,\Omega_f} + \langle\sigma(w_u).\nu, z(u)\rangle \ge C|u|^2_{1,\Omega_f}. \tag{3.19}$$

Hence, surjectivity of $\lambda I + A + L$ into V' for λ sufficiently large and ϵ sufficiently small follows. Since $f - A(0,w_{gh}) \in V'$, there exists $u \in V$ satisfying the above system, and therefore by the above there exists $w = w(u,g,h) = w_{gh} + w_u \in H^1$ satisfying the system. Moreover, a solution z can be reconstructed from the equation $z - \lambda w = g$. In other words, given $f,g,h \in \mathcal{H}$ we have a solution $[u,w,z] \in \mathcal{H}$ such that $\lambda Y - \mathcal{A}_L Y = [f,g,h]$. Therefore, ω m-accretivity of the operator

$-\mathcal{A}_L$ is established. Now that we have established that $-\mathcal{A}_L$ is ω m-accretive, $\mathcal{A}_L$ generates a contraction semigroup on $\mathcal{H}$. In other words, there exists a semigroup solution $[u, w, w_t] \in C([0,T];\mathcal{H})$ (resp. $C([0,T];\mathcal{D}(\mathcal{A}_L))$) to the system (3.6), when $[u_0, w_0, w_1] \in \mathcal{H}$ (resp. $\mathcal{D}(\mathcal{A}_L)$). □

Our next step is to establish variational formulation for the auxiliary problem (3.1). This is given in the corollary below.

COROLLARY 3.1. Let $[u_0, w_0, w_1] \in D(A_L)$, then the semigroup solution $Y = [u, w, w_t]$, obtained in Lemma 3.1 satisfies the following regularity properties

- $Y \in C([0,T];\mathcal{H})$, $Y_t \in L_\infty([0,T];\mathcal{H})$
- $A_L(u, \sigma(w)) \in C([0,T];\mathcal{H})$, $\operatorname{div} \sigma(w) \in L_2(\Omega_s)$
- $w_t|_{\Gamma_s} = u|_{\Gamma_s} \in C([0,T]; H^{1/2}(\Gamma_s))$, $\sigma(w) \cdot \nu \in C([0,T]; H^{-1/2}(\Gamma_s))$

Moreover, $Y(t)$ satisfies for all $t \in [0,T]$, the variational form of *Problem-L* given in (3.1), (3.2) and (3.3).

PROOF. By the nonlinear semigroup theory we know that solutions corresponding to initial data in $D(\mathcal{A}_L)$ remain continuous with the values in $D(\mathcal{A}_L)$. This, in particular implies that $\operatorname{div}(\sigma(w)) \in L_2(\Omega_s)$. Moreover, $w \in H^1(\Omega_s)$. Hence, using Remark 3.1 one infers that $\sigma(w) \cdot \nu \in C([0,T]; H^{-1/2}(\Gamma_s))$. Going back to equation (3.6) and taking appropriate inner product with test functions $\phi \in V$ and $\psi, \xi \in H^1(\Omega_s)$ leads to:

$$\begin{aligned} (u_t, \phi) + (A_L(u, \sigma(w)), \phi) &= 0 \\ (w_{tt}, \psi) - (\operatorname{div} \sigma(w), \psi) &= 0 \\ w_t|_{\Gamma_s} = u|_{\Gamma_s}, &\, in\ C([0,T]; H^{1/2}(\Gamma)) \end{aligned} \tag{3.20}$$

for all $\phi \in V, \psi, \xi \in H^1(\Omega_s)$. Using explicit representation of A_L in the first equation and integrating by parts the second equation , while recalling that $\sigma(w) \cdot \nu \in H^{-1/2}(\Gamma_s)$ completes the proof of the fact that $Y(t)$ satisfies the variational form of the equations (3.1) and (3.3). □

The final result of this section is the claim that the semigroup solution corresponding to the initial data in $\mathcal{H}$ also satisfies the variational form of the equation, and hence it is "weak" solution according to our terminology. The main obstacle in obtaining this result is the lack of a priori regularity of traces $\sigma(w).\nu$ defined for finite energy solutions.

In fact, while regularity of $\sigma(w) \cdot \nu$ on the boundary results from trace theory applicable to *elliptic* problems in the case of solutions evolving in the domain of the generator $\mathcal{A}_L$; this elliptic regularity is not available in the case of finite energy solutions,. What is needed is the corresponding *hyperbolic* regularity. Thus, in order to cope with this we shall develop an appropriate trace theory for these solutions. The final result claimed, which extends the result of the Corollary above to all finite energy solutions is the following:

THEOREM 3.2. **Weak solution for Problem-L** *Let $[u_0, w_0, w_1] \in \mathcal{H}$ and let $Y = [u, w, w_t]$ be the corresponding semigroup solution given by Lemma 3.1. Then, the following regularity result holds:*

- $Y = [u, w, w_t] \in C([0,T];\mathcal{H}), u \in L_2([0,T];V)$
- $w_t|_{\Gamma_s} \in L_2([0,T]; H^{1/2}(\Gamma_s))$
- $\sigma(w) \cdot \nu \in L_2([0,T]; H^{-1/2}(\Gamma_s))$

Moreover, the semigroup solution obtained in lemma 3.1 satisfies a.e in time $t \in (0,T)$ *the variational form of Problem-L given in (3.1) and (3.3).*

In order to prove Theorem 3.2 we will need to develop some trace results valid for weak solutions to the linear wave equation. These are given below.

THEOREM 3.3. *Consider the linear wave equation.*

$$\left\{\begin{array}{rl} w_{tt} - \operatorname{div}\sigma(w) = 0 & \Omega_s \times [0,T] \\ w_t|_{\Gamma_s} = g & \in L_2([0,T]; H^{1/2}(\Gamma_s)) \\ w(0,.) = w_0 & \in H^1(\Omega_s) \\ w_t(0,.) = w_1 & \in L_2(\Omega_s) \end{array}\right. \tag{3.21}$$

Then the map:

$$[w_0, w_1, g] \rightarrow [w, w_t, \sigma(w).\nu]$$

is bounded from:

$$H^1(\Omega_s) \times L_2(\Omega_s) \times L_2([0,T]; H^{1/2}(\Gamma_s))$$

$$\text{to } C([0,T]; H^1(\Omega_s)) \times C([0,T]; L_2(\Omega_s)) \times L_2([0,T]; H^{-1/2}(\Gamma_s))$$

The proof of this theorem is given in section 7. Here we only note that the trace regularity postulated above does not follow from the interior regularity of solutions to wave equation. This is the so called "hidden" regularity reminiscent of "hidden" regularity results in **[22]**.

PROOF. of Theorem 3.2 From the corollary, we already know that all the assertions postulated in the theorem are valid for solutions originating in $\mathcal{D}(\mathcal{A}_L)$. We claim that $\mathcal{D}(\mathcal{A}_L)$ is dense in $\mathcal{H}$

For, consider $Y - \frac{1}{\lambda}\mathcal{A}_L Y$ which is surjective for a given L by lemma 3.1. So, given $[f,g,h] \in \mathcal{H}$, there exists $Y_{L,\lambda} \in \mathcal{D}(\mathcal{A}_L)$ such that $Y_{L,\lambda} - \frac{1}{\lambda}\mathcal{A}_L Y_{L,\lambda} = [f,g,h]$. Letting λ go to ∞, we have $lim_{\lambda\to\infty} Y_{L,\lambda} = [f,g,h]$. This proves density of $\mathcal{D}(\mathcal{A}_L)$ in $\mathcal{H}$.

Therefore, each initial data $Y_0 \in \mathcal{H}$ can be approximated (in strong topology) by initial data $Y_n \in \mathcal{D}(\mathcal{A}_L)$. The corresponding solutions converge strongly in the topology of $\mathcal{H}$. (the latter follows from the definition of a semigroup solution). In addition, writing the equation (3.1), (3.3) for the difference of two smooth solutions and applying the standard energy methods by taking inner product of the equations with this difference and using properties of L in (3.1) we obtain:

$$u_n \rightarrow u, \ in \ L_2([0,T]; V)$$

Regularity results on linear waves allow us to conclude:

$$\begin{aligned} |\tilde{w}(t,.)|_{1,\Omega_s} + |\tilde{w}_t(t,.)|_{0,\Omega_s} + |\sigma(\tilde{w}).\nu|^2_{L_2([0,T];H^{-1/2}(\Gamma_s))} \\ \leq C[|\tilde{w}_0|_{1,\Omega_s} + |\tilde{w}_1|_{0,\Omega_s} + |\tilde{u}|_{L_2([0,T];H^{1/2}(\Gamma_s))}] \end{aligned} \tag{3.22}$$

where $\tilde{w} \equiv w_n - w_m$ with similar notation applied to other quantities. This implies the following strong convergence on the boundary, that is critical for the argument:

$$\sigma(w_n) \cdot \nu \rightarrow \sigma(w) \cdot \nu, \ in \ L_2([0,T]; (H^{-1/2}(\Gamma_s))$$

The above convergence allows passing with the limit on the variational form written first for smooth solutions $Y_n(t)$. This results in a variational form (3.1), (3.2), and (3.3) satisfied by $[u, w, w_t]$ given $[u_0, w_0, w_1] \in \mathcal{H}$. □

3.2. Approximations of the nonlinear term. Let us denote by $B(.,.) : V \times V \to V'$ the operator defined by:

$$(B(y,x),z) \equiv ((y.\nabla)x,z) - \frac{1}{2}\langle (y.\nu)x,z\rangle \tag{3.23}$$

and denote by $B(.) : V \to V'$ the operator defined by:

$$B(u) \equiv B(u,u) \tag{3.24}$$

Also, denote by $b(u,v,w)$ the trilinear form :

$$b(u,v,w) \equiv (B(u,v),w) \equiv ((u.\nabla)v,w) - \frac{1}{2}\langle (u.\nu)v,w\rangle \tag{3.25}$$

One can clearly see that $b(u,v,w)$ is indeed trilinear. We demonstrate some properties of $b(.,.,.)$ through the following lemma. In what follows, we shall use the following notation for the H^s norm on Ω_f:

$$|u|_s = |u|_{s,\Omega_f},\ |u|_0 = |u|_{L_2(\Omega_f)}$$

LEMMA 3.4. *The trilinear mapping b satisfies the following estimates for all $u \in V$, $v,w \in H^1(\Omega_f)$ with dimension of $\Omega = n$:*

(1)
$$|((u.\nabla)v,w)| \le C|u|_{s_1}|v|_{s_2+1}|w|_{s_3}$$
where $s_1,s_2,s_3 \ge 0$ and $s_1+s_2+s_3 \ge \frac{n}{2}$ if $s_i \ne \frac{n}{2}$ $\forall i = 1,2,3$ or $s_1+s_2+s_3 > \frac{n}{2}$ if $s_i = \frac{n}{2}$ for some $i = 1,2,3$.

(2)
$$\langle |(u.\nu)v,w\rangle| \le C|u|_{s_1}|v|_{s_2}|w|_{s_3}$$
where $s_1 \ge 1/2$, $s_2,s_3 > 1/2$ and $s_1+s_2+s_3 \ge \frac{n+2}{2}$

(3) *If $n=2$*
$$|b(u,v,w)| \le C_1|u|_{1/2}|v|_1|w|_{1/2} + C_2|u|_{1/2}|v|_{3/4}|w|_{3/4}$$

(4) *If $n=3$*
$$|b(u,v,w)| \le C|u|_{1/2}|v|_1|w|_1$$
$$|b(u,v,w)| \le C|u|_{3/4}|v|_1|w|_{3/4}$$

(5) *$b(u,v,w)$= - $b(u,w,v)$, $b(u,u,u) = 0$*

PROOF. (1) This is a standard estimate of the nonlinear term of the Navier Stokes equations. See [**26, 9**] .

(2) Assume s_i as in hypothesis and let $s_i = m_i + 1/2$ and consider the following cases:

case 1. Each $m_i < \frac{n-1}{2}$, then we let $\frac{1}{q_i} = \frac{1}{2} - \frac{m_i}{n-1}$. Notice that given the hypothesis on s_i, $\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} \le 1$ and thus we estimate the nonlinear term in question as follows:

$$|\langle (u.\nu)v,w\rangle| \le |(u.\nu)|_{L_{q_1}(\Gamma_s)}|v|_{L_{q_2}(\Gamma_s)}|w|_{L_{q_3}(\Gamma_s)} \tag{3.26}$$

We next apply the Sobolev imbedding theorem with dimension of $\Gamma_s = n-1$, which states that if $\frac{1}{p} - \frac{m}{n-1} > 0$ then $W^{m,p} \subset L^q$ continuously where $\frac{1}{q} = \frac{1}{p} - \frac{m}{n-1}$ for $m < \frac{n-1}{2}$. In our case of $p=2$, we then conclude that $H^{m_i}(\Gamma_s) \subset L_{q_i}(\Gamma_s)$ and thus:

(3.27) $$|\langle (u.\nu)v, w\rangle| \leq C|u.\nu|_{m_1,\Gamma_s}|v|_{m_2,\Gamma_s}|w|_{m_3,\Gamma_s}$$

case 2. If for some i say m_1, $m_1 > \frac{n-1}{2}$ we can take $q_1 = \infty$ instead and q_2 q_3 as before so that (3.26) applies. We then use the Sobolev imbedding for the case $m_1 > \frac{n-1}{2}$ in which $H^{m_1}(\Gamma_s) \subset L^\infty(\Gamma_s)$. Therefore, (3.27) follows.

case 3. If $m_1 = \frac{n-1}{2}$, we take q_2 and q_3 as before and notice that $\frac{1}{q_2}$ and $\frac{1}{q_3}$ can not both equal to 1/2 since both $m_2, m_3 > 0$, then $\frac{1}{q_2} + \frac{1}{q_3} < 1$, and we then use the Sobolev imbedding result for this case: $H^{m_1}(\Gamma_s) \subset \bigcap_{q>2} L_q(\Gamma_s) \subset L_{q_1}(\Gamma_s)$, where we let $\frac{1}{q_1} = 1 - \frac{1}{q_2} - \frac{1}{q_3}$ So, the inequality (3.26) and consequently (3.27) still hold.

We next apply the continuity of the trace map from $H^{m_i+1/2}(\Omega_f) \to H^{m_i}(\Gamma_s)$ which holds for $m_i > 0$ to obtain from (3.27) the following estimate: $|\langle (u.\nu)v, w\rangle| \leq C|u|_{m_1+1/2}|v|_{m_2+1/2}|w|_{m_3+1/2}$. On the other hand if $m_1 = 0$ we can still apply the continuity of the trace map $\gamma_\nu(u) = u.\nu$ from $H \to H^{-1/2}(\Gamma_s)$.

Therefore the result follows under the desired conditions on the exponents s_i: $|\langle (u.\nu)v, w\rangle| \leq C|u|_{s_1}|v|_{s_2}|w|_{s_3}$

Hence, the desired estimate applies to the mapping b.

(3) The result follows from combining the preceding two results with $s_1 = 1/2$, $s_2 = 0$ $s_3 = 1/2$ for the interior inner product and $s_1 = 1/2$, $s_2 = 3/4$ and $s_3 = 3/4$.
(4) The result similarly follows from part (1) and (2).
(5) Using Einstein summation convention:

$$b(u,v,w) = \int_{\Omega_f} u_i(D_i v_j)w_j - \frac{1}{2}\int_{\Gamma_s}(u_i\nu_i)v_j w_j$$
$$= \int_{\Omega_f} u_i D_i(v_j w_j) - \int_{\Omega_f} u_i(D_i w_j)v_j - \frac{1}{2}\int_{\Gamma_s} u_i\nu_i v_j w_j$$
$$= -\int_{\Omega_f} D_i u_i(v_j w_j) + \int_{\Gamma_s} u_i\nu_i v_j w_j - \int_{\Omega_f} u_i(D_i w_j)v_j - \frac{1}{2}\int_{\Gamma_s} u_i\nu_i v_j w_j$$
$$= -(\operatorname{div} u, vw) + \frac{1}{2}\langle (u.\nu)w, v\rangle - ((u.\nabla)w, v)$$
$$= 0 - b(u,w,v) = -b(u,w,v)$$

(6) $|b(u,u,u)| = 0$ follows immediately from the preceding result.

□

Now, let us consider the approximate problem to the system (2.1), that contains a truncation of the nonlinear term $B(u)$ satisfying the criteria for the L operator. This will enable us to apply lemma 3.1. Following [5] we define $B_n : V \to V'$ as the operator:

(3.28) $$B_n(u) \equiv \begin{cases} B(u), & |u|_1 \leq n \\ n^2 \frac{B(u)}{|u|_1^2} & |u|_1 \geq n \end{cases}$$

We note that $|B_n(u)|_{V'} \leq |B(u)|_{V'}$. In what follows we shall denote by $\mathcal{A}_n$ the operator $\mathcal{A}_L$ with $L(u)$ replaced by $B_n(u)$.

LEMMA 3.5.
- *Let* $[u_0, w_0, w_1] \in \mathcal{H}$. *Then , for each value of* n $\mathcal{A}_n$ *generates strongly continuous nonlinear semigroup on* $\mathcal{H}$.
- *Let* $[u_0, w_0, w_1] \in \mathcal{H}$. *Then semigroup solutions* $[u^n, w^n, w_t^n]$ *satisfy the variational form*

$$(u_t^n, \phi) + (\epsilon(u^n), \epsilon(\phi)) + (B_n u^n, \phi) + \langle \sigma(w^n).\nu, \phi\rangle = 0 \tag{3.29}$$

$$w_t^n|_{\Gamma_s} = u^n|_{\Gamma_s},\ in\ C([0,T]; H^{\frac{1}{2}}(\Gamma_s)),\ \sigma(w^n)\nu \in L_2(0T.H^{-1/2}(\Gamma_f)) \tag{3.30}$$

$$(w_{tt}^n, \psi) + (\sigma(w^n), \epsilon(\psi)) - \langle \sigma(w^n).\nu, \psi\rangle = 0 \tag{3.31}$$

with test functions $\phi \in V$ *and* $\psi \in H^1(\Omega)$.

PROOF. It suffices to apply Theorem 3.2. To this end, we verify that B_n indeed satisfies the criteria placed on L in (3.1).

Note that we will only use trilinearity of b and the properties of b from lemma 3.4 in the proofs below and that we shall use estimates of the nonlinear term b for the three dimensional case which is also applicable in the two dimensional case as well:

B_n**, for each** n **, is Locally Lipschitz from** $V \to V'$: We will prove that B_n is Lipschitz on a ball centered at 0 of radius R in V. Let $u, v \in B_V(0, R)$

case 1 : $|u|_1 \le n$, $|v|_1 \le n$

$$\begin{aligned}(B_n u - B_n v, w) &= b(u,u,w) - b(v,v,w)\\ &= b(u, u-v, w) - b(v-u, v, w)\\ &\le C|u|_{\frac{1}{2}}|u-v|_1|w|_1 + C|u-v|_{\frac{1}{2}}|v|_1|w|_1\\ &\le C|u-v|_1|w|_1(|u|_1 + |v|_1) \le C(R,n)|u-v|_1\end{aligned}$$

Taking the supremum of both sides over all $w \in V$ of unit norm yields the desired conclusion with the Lipschitz constant $2Cn$.

case 2: $|u|_1 \ge n$, $|v|_1 \ge n$

$$(B_n u - B_n v, w) = n^2[b(u,u,\frac{w}{|u|_1^2}) - b(v,v,\frac{w}{|v|_1^2})]$$

$$= n^2[b(u, u-v, \frac{w}{|u|_1^2}) + b(u-v, v, \frac{w}{|u|_1^2}) - b(v, v, w(\frac{1}{|v|_1^2} - \frac{1}{|u|_1^2}))]$$

$$\le n^2 C[|u|_{\frac{1}{2}}|u-v|_1\frac{|w|_1}{|u|_1^2} + |u-v|_{\frac{1}{2}}|v|_1\frac{|w|_1}{|u|_1^2} + |v|_{\frac{1}{2}}|v|_1|w|_1(\frac{1}{|v|_1^2} - \frac{1}{|u|_1^2})]$$

$$\le C|w|_1|u-v|_1(|u|_1 + |v|_1) + Cn^2|w|_1|v|_1^2(\frac{1}{|v|_1^2} - \frac{1}{|u|_1^2})]$$

$$\le C|w|_1|u-v|_1(|u|_1 + |v|_1) + Cn^2|w|_1|v|_1^2(\frac{|u|_1^2 - |v|_1^2}{|v|_1^2|u|_1^2})$$

$$\le C|w|_1|u-v|_1(|u|_1 + |v|_1) + C|w|_1(|u|_1 - |v|_1)(|u|_1 + |v|_1) \le C(n,R)|u-v|_1$$

Taking supremum over all w with unit norm yields the desired conclusion with Lipschitz constant of $2C$

case 3: $|u|_1 \ge n$, $|v|_1 \le n$

$$(B_n u - B_n v, w) = n^2 b(u,u,\frac{w}{|u|_1^2}) - b(v,v,w)$$

$$= n^2[b(u, u-v, \frac{w}{|u|_1^2}) + b(u-v, v, \frac{w}{|u|_1^2})] - b(v, v, w(1 - \frac{n^2}{|u|_1^2}))]$$

$$\leq n^2 C[|u|_{\frac{1}{2}}|u-v|_1 \frac{|w|_1}{|u|_1^2} + |u-v|_{\frac{1}{2}}|v|_1 \frac{|w|_1}{|u|_1^2}] + C|v|_{\frac{1}{2}}|v|_1|w|_1(1-\frac{n^2}{|u|_1^2})]$$

$$\leq C|w|_1|u-v|_1(|u|_1+|v|_1) + C|w|_1(|u|_1^2-|v|_1^2)$$

$$\leq C|w|_1|u-v|_1(|u|_1+|v|_1) + C|w|_1(|u|_1-|v|_1)(|u|_1+|v|_1) \leq C(n,R)|u-v|_1$$

Again taking supremum over all unit vectors $w \in V$ yields the desired conclusion.

B_n **satisfies condition (3.4)** :

Case 1: $|y|_1 \leq n$, $|z|_1 \leq n$

In what follows we shall set $s_1 = 3/41$, $s_2 = 1$ and $s_3 = 3/4$ to estimate the nonlinear term $b(.,.,.)$ as in lemma 3.4)(4).

$$|(B_n y - B_n z, y-z)| = |b(y,y,y-z) - b(z,z,y-z)|$$
$$= |b(y-z,y,y-z) + b(z,y-z,y-z)|$$
$$\leq C|y-z|_1^{3/4}|y-z|_0^{1/4}|y-z|_1(|y|_1+|z|_1) \leq 2Cn|y-z|_1^{7/4}|y-z|_0^{1/4}$$
$$\leq \epsilon(n)|y-z|_1^2 + C_{\epsilon(n)}|y-z|_0^2$$

Where the last inequality follows from Young's inequality. Again taking supremum over all unit vectors $\phi \in V$ yields the desired conclusion.

Case 2: $|y|_1 \geq n$, $|z|_1 \geq n$ and WLOG $|y|_1 \geq |z|_1$

$$|(B_n y - B_n z, y-z)| = |\frac{n^2}{|y|_1^2}(b(y,y,y-z) - b(z,z,y-z)) + (\frac{n^2}{|y|_1^2} - \frac{n^2}{|z|_1^2})b(z,z,y-z)|$$

$$= |\frac{n^2}{|y|_1^2}[b(y-z,y,y-z) + b(z,y-z,y-z)] + n^2(\frac{|z|_1^2-|y|_1^2}{|y|_1^2|z|_1^2})b(z,z,y-z)|$$

$$\leq C\frac{n^2}{|y|_1^2}|y-z|_1|y-z|_1^{3/4}|y-z|_0^{1/4}[|y|_1+|z|_1] + C(\frac{n^2}{|y|_1^2|z|_1^2})||z|_1^2-|y|_1^2||z|_1^2|y-z|_{3/4}$$

$$\leq 2C\frac{n^2}{|y|_1}|y-z|_1^{7/4}|y-z|_0^{1/4} + C(\frac{n^2}{|y|_1^2|z|_1^2})||z|_1-|y|_1|(|z|_1+|y|_1)|z|_1^2|y-z|_{3/4}$$

$$\leq 2Cn|y-z|_1^{7/4}|y-z|_0^{1/4} + C\frac{n^2}{|y|_1^2}|z-y|_1(|z|_1+|y|_1)|y-z|_{3/4}$$

$$\leq 2Cn|y-z|_1^{7/4}|y-z|_0^{1/4} + 2C\frac{n^2}{|y|_1}|z-y|^{7/4}|y-z|_0^{1/4}$$

$$\leq 4Cn|y-z|_1^{7/4}|y-z|_0^{1/4}$$
$$\leq \epsilon(n)|y-z|_1^2 + C_{\epsilon(n)}|y-z|_0^2$$

where again we used Young's inequality to obtain the desired conclusion.

Case 3: $|y|_1 \geq n$, $|z|_1 \leq n$

$$|(B_n y - B_n z, y-z)| = |\frac{n^2}{|y|_1^2}b(y,y,y-z) - b(z,z,y-z)|$$

$$\leq |\frac{n^2}{|y|_1^2} - 1||b(z,z,y-z)| + \frac{n^2}{|y|_1^2}|b(y,y,y-z) - b(z,z,y-z)|$$

We now use the same estimate from case 1 above for the term $b(y,y,y-z) - b(z,z,y-z)$ to obtain:

$$|(B_n y - B_n z, y-z)| \leq C\frac{|y|_1^2-n^2}{|y|_1^2}|z|_1^2|y-z|_1^{3/4}|y-z|_0^{1/4} + 2Cn|y-z|_1^{7/4}|y-z|_0^{1/4}$$

$$\leq C\frac{|y|_1^2-|z|_1^2}{|y|_1}|z|_1|y-z|_1^{3/4}|y-z|_0^{1/4}+2Cn|y-z|_1^{7/4}|y-z|_0^{1/4}$$

$$\leq C\frac{n}{|y|_1}(|y|_1-|z|_1)(|y|_1+|z|_1)|y-z|_1^{3/4}|y-z|_0^{1/4}+2Cn|y-z|_1^{7/4}|y-z|_0^{1/4}$$

$$\leq 2Cn|y-z|_1|y-z|_1^{3/4}|y-z|_0^{1/4}+2Cn|y-z|_1^{7/4}|y-z|_0^{1/4}$$

$$\leq 4Cn|y-z|_1^{7/4}|y-z|_0^{1/4}\leq \epsilon(n)|y-z|_1^2+C_{\epsilon(n)}|y-z|_0^2$$

where again we used Young's inequality to obtain the desired conclusion. □

3.3. Weak Solution to the Fully Nonlinear Interaction.

THEOREM 3.6. *For every initial condition* $[u_0, w_0, w_1] \in \mathcal{H}$ *there exists a solution* $[u, w, w_t] \in C([0,T];\mathcal{H})$ *such that* $u \in L_2([0,T];V)$, Bu, $\frac{du}{dt} \in L_1([0,T],V')$, $\nabla w|_{\Gamma_s} \in L_2([0,T];H^{-\frac{1}{2}}(\Gamma_s))$ *and* $w_t|_{\Gamma_s} \in L_2([0,T];H^{\frac{1}{2}}(\Gamma_s))$ *satisfying* $\forall\psi \in H^1(\Omega_s)$, $\forall\phi \in V$ *a.e.* t :

$$(u_t,\phi)+(\epsilon(u),\epsilon(\phi))+(Bu,\phi)+\langle\sigma(w).\nu,\phi\rangle=0 \tag{3.32}$$

$$w_t|_{\Gamma_s}=u|_{\Gamma_s},\ in\ L_2([0,T],H^{\frac{1}{2}}(\Gamma_s)) \tag{3.33}$$

$$(w_{tt},\psi)+(\sigma(w),\epsilon(\psi))-\langle\sigma(w).\nu,\psi\rangle=0 \tag{3.34}$$

PROOF. From lemma 3.5, we can conclude that there exists a solution of the approximate problem $[u_n, w_n, w_{nt}] \in C([0,T];\mathcal{H})$ satisfying (3.29), (3.30) and (3.31):

$$(u_{nt},\phi)+(\epsilon(u_n),\epsilon(\phi))+(B_nu_n,\phi)+\langle\sigma(w_n).\nu,\phi\rangle=0\ \forall\phi\in V \tag{3.35}$$

$$(w_{ntt},\psi)+(\sigma(w_n),\epsilon(\psi))-\langle\sigma(w_n).\nu,\psi\rangle=0\ \forall\psi\in H^1(\Omega_s) \tag{3.36}$$

$$w_{nt}|_{\Gamma_s}=u_n|_{\Gamma_s},\ in\ L_2([0,T],H^{\frac{1}{2}}(\Gamma_s))$$

We shall obtain the desired conclusion by passing through the limit in n. This requires several a priori bounds. In addition, for smooth initial data the solution u^n is smooth, which allows for the application of energy methods.

Step 1: A priori bounds

·To derive bounds for u_n, consider the system (3.35) with $\phi = u_n$ and (3.36) with $\psi = w_{nt}$, and use the result in lemma 3.4 that $b(u_n, u_n, u_n) = 0$ to obtain:

$$\frac{1}{2}\frac{d}{dt}|u_n|_{0,\Omega_f}^2+|\epsilon(u_n)(t)|_{0,\Omega_f}^2+\langle\sigma(w_n).\nu,u_n\rangle=0$$

$$\frac{1}{2}\frac{d}{dt}(|w_{nt}|_{0,\Omega_s}^2)+(\sigma(w_n),\epsilon(w_{nt}))-\langle\sigma(w_n).\nu,w_{nt}\rangle=0$$

Now, using the boundary condition $w_{nt}|_{\Gamma_s} = u_n|_{\Gamma_s}$ we can combine the two equations into one via cancelation of transmission conditions.

$$\frac{1}{2}\frac{d}{dt}|u_n|_{0,\Omega_f}^2+|\epsilon(u_n)(t)|_{0,\Omega_f}^2+\frac{1}{2}\frac{d}{dt}(|w_{nt}|_{0,\Omega_s}^2+(\sigma(w),\epsilon(w))_{0,\Omega_s})\leq 0$$

Integrating in time we obtain for all $0 \leq t \leq T$:

$$|u_n(t)|_{0,\Omega_f}^2+2\int_0^t|u_n(s)|_{1,\Omega_f}^2ds+|w_{nt}(t)|_{0,\Omega_s}^2+(\sigma(w_n)(t),\epsilon(w_n)(t))$$
$$\leq|u_0|_{0,\Omega_f}^2+(\sigma(w_0),\epsilon(w_0))+|w_1|_{0,\Omega_s}^2$$

- **u a priori bounds**

We have the following estimates:

$$|u_n(t)|^2_{0,\Omega_f} \leq |u_0|^2_{0,\Omega_f} + (\sigma(w_0), \epsilon(w_0))_{0,\Omega_s} + |w_1|^2_{0,\Omega_s} = E(0) \tag{3.37}$$

$$\int_0^T |u_n(t)|^2_{1,\Omega_f} dt \leq E(0) \tag{3.38}$$

- **w, w_t a priori bounds**

We also obtain from the above $\forall t \in [0,T]$ and $\forall n$;

$$|w_{nt}(t,.)|^2_{0,\Omega_s} \leq E(0) \tag{3.39}$$

$$(\sigma(w_n)(t), \epsilon(w_n)(t))_{0,\Omega_s} \leq E(0) \tag{3.40}$$

In addition, notice

$$|w_n(t,.)|^2_{1,\Omega_s} = (\sigma(w_n)(t), \epsilon(w_n)(t)) + |w_n(t)|^2_{0,\Omega_s}$$

$$\leq E(0) + |\int_0^t w_{nt}(s)ds + w_0|^2_{0,\Omega_s} \leq E(0) + 2T\int_0^t |w_{nt}|^2_{0,\Omega_s} ds + 2|w_0|^2_{0,\Omega_s}$$

$$\leq C_T E(0)$$

The last inequality follows from (3.39) and (3.40).

Hence

$$|w_n(t,.)|^2_{1,\Omega_s} \leq C_T E(0) \tag{3.41}$$

- **$\langle \sigma w, \nu \rangle$ a priori bounds**

Before proceeding to derive the estimates necessary for passing through the limit, we shall invoke the trace regularity result in Theorem 3.3 applied with $g = u_n|_{\Gamma_s} \in L_2([0,T]; H^{\frac{1}{2}}(\Gamma_s))$, which implies the following estimates on the wave components $w_n, w_{nt}, \sigma(w_n).\nu$ for all $t \in [0,T]$, and for all n:

$$|\sigma(w_n).\nu|^2_{L_2([0,T];H^{-\frac{1}{2}}(\Gamma_s))} \leq C[|w_0|^2_{1,\Omega_s} + |w_1|^2_{0,\Omega_s} + |u_n|^2_{L_2([0,T];H^{\frac{1}{2}}(\Gamma_s))}] \tag{3.42}$$

Therefore, using continuity of the trace from $V \to H^{1/2}(\Gamma_s)$ and the estimate (3.38) we have that:

$$|\sigma(w_n).\nu|^2_{L_2([0,T];H^{-\frac{1}{2}}(\Gamma_s))} \leq C(|u_n|^2_{L_2([0,T];V)} + E(0)) \leq CE(0) \tag{3.43}$$

- **u_t a priori bounds**

We shall also need a bound for $\frac{d}{dt}u_n$. Let us consider (3.35) with ϕ time independent:

$$(\frac{d}{dt}u_n, \phi) = -(\epsilon(u_n), \epsilon(\phi)) - (B_n u_n, \phi) - \langle \sigma(w_n).\nu, \phi \rangle \tag{3.44}$$

We next estimate a uniform bound on the norm of the time derivative u_t in V' by taking the supremum over all $\phi \in V$ of unit norm:

$$|\frac{d}{dt}u_n|_{V'} \leq C[|u_n(t)|_{1,\Omega_f} + |B_n u_n(t)|_{V'} + |\sigma(w_n)(t).\nu|_{-\frac{1}{2},\Gamma_s}] \tag{3.45}$$

We finally apply the estimates of the nonlinear term b with $n = 3$ from lemma 3.4(4) with $s_1 = 1/2, s_2 = s_3 = 1$ and interpolate between $H^1(\Omega_f)$ and $L_2(\Omega_f)$:

$$|B_n u_n|_{V'} \leq |Bu_n|_{V'} = \sup_{|\phi|_{1,\Omega_f}=1} b(u,u,\phi) \leq |u_n|^{\frac{1}{2}}_{0,\Omega_f} |u_n|^{3/2}_{1,\Omega_f} \tag{3.46}$$

We next estimate the norm of derivative in $L_2([0,T];V')$ using (3.45) and (3.46) to obtain

$$(\int_0^T |\frac{d}{dt}u_n(s)|^{4/3}_{V'}ds)^{3/4}$$
$$\leq \int_0^T |u_n(s)|^{4/3}_{1,\Omega_f}ds]^{3/4} + [\int_0^T |B_n u_n(s)|^{4/3}_{V'}ds]^{3/4} + [\int_0^T |\sigma(w_n)(s).\nu|^{4/3}_{-\frac{1}{2},\Gamma_s}ds]^{3/4}$$
$$\leq [\int_0^T |u_n(s)|^2_{1,\Omega_f}ds]^{\frac{1}{2}} T^{1/4} + [\int_0^T |B_n u_n(s)|^{4/3}_{V'}ds]^{3/4} + [\int_0^T |\sigma(w_n)(s).\nu|^2_{-\frac{1}{2},\Gamma_s}ds]^{\frac{1}{2}} T^{1/4}$$
$$\leq E^{1/2}(0)T^{1/4} + \sup_{t\in[0,T]} |u_n(t)|^{\frac{1}{2}}_{0,\Omega_f} [\int_0^T |u_n(s)|^2_{1,\Omega_f}ds]^{3/4} + |\sigma(w_n)(s).\nu|_{L_2([0,T];H^{-\frac{1}{2}}(\Gamma_s))} T^{1/4}$$
$$\leq C_T E^{1/2}(0) + M_T E(0)$$

The final estimate above follows from the uniform boundedness of u_n in $L_\infty([0,T];H)$ from (3.37) and in $L_2([0,T];V)$ as given in (3.38), as well as the bound in (3.43). Therefore $\frac{du_n}{dt}$ are uniformly bounded in $L^{4/3}([0,T];V')$.

$$(\int_0^T |\frac{d}{dt}u_n(s)|^{4/3}_{V'}ds)^{3/4} \leq C_T E^{1/2}(0) + M_T E(0) \tag{3.47}$$

The 4/3 regularity here is the best one can achieve for dimension of Ω of $n = 3$, given the bound in (3.46), by utilizing uniform integrability of $|u_n(s)|^2_{1,\Omega_f}$.

Step 2: Passing through the limit in the w equation

- From the bounds (3.39) w_{nt} is uniformly bounded in $L_\infty([0,T];L_2(\Omega_s))$, so we can extract a weak* convergent subsequence of w_{nt} in $L_\infty([0,T];L_2(\Omega_s))$ to $w_t \in L_\infty([0,T];L_2(\Omega_s))$.
- ¿From (3.41), w_n is uniformly bounded in $L_\infty([0,T];H^1(\Omega_s))$. Therefore, we can extract a weak* convergent subsequence of w_n in $L_\infty([0,T];H^1(\Omega_s))$ to $w \in L_\infty([0,T];H^1(\Omega_s))$.
- By the uniform bound on $\sigma.\nu(w)$ in (3.43) we can also extract a subsequence of $\sigma(w_n).\nu$ converging to some l weakly in $L_2([0,T];H^{-\frac{1}{2}}(\Gamma_s))$.

We now pass through the limit in (3.36). By weak convergence of $w_n \to w$ in $L_2([0,T];H^1(\Omega_s))$ (a.e. t weak* in $H^1(\Omega_s)$) we have

$$(\sigma(w_n),\epsilon(\psi)) \to (\sigma(w),\epsilon(\psi)), a.e\ in\ t \tag{3.48}$$

We next appeal to weak convergence of $w_{ntt} \to w_{tt}$ in $H^{-1}([0,T];L_2(\Omega_s))$ which follows from weak convergence of $w_{nt} \to w_t$ in $L_2([0,T];L_2(\Omega_s))$.

$$\frac{d}{dt}(w_{nt},\psi) \to \frac{d}{dt}(w_t,\psi), \forall\psi \in H^1(\Omega_s) \tag{3.49}$$

By weak convergence of $\sigma(w_n).\nu$ in $L_2([0,T];H^{-\frac{1}{2}}(\Gamma_s))$ to some $l \in L_2(0T, H^{-1/2}(\Gamma_s))$, we also have

$$\langle \sigma(w_n).\nu, \psi \rangle \to \langle l, \psi \rangle \text{ in } L_2([0,T]), \forall \psi \in H^1(\Omega_s) \tag{3.50}$$

Hence, we collectively obtain the following equation satisfied by w

$$\frac{d}{dt}(w_t, \psi) + (\sigma(w), \epsilon(\psi)) - \langle l, \psi \rangle = 0 \text{ in } H^{-1}([0,T]) \tag{3.51}$$

On the other hand, using Green's theorem, equation (3.51) can be reexpressed for all $\psi \in H_0^1(\Omega_s)$ as

$$(w_{tt}, \psi) - (\operatorname{div} \sigma(w), \psi) = 0$$

Thus, $w_{tt} = \operatorname{div} \sigma(w)$ in $L_\infty([0,T];H^{-1}(\Omega))$. Moreover, since $w_{tt} \in H^{-1}([0,T];L_2(\Omega_s))$, $\operatorname{div} \sigma(w) \in H^{-1}([0,T];L_2(\Omega_s))$. Hence

$$(\operatorname{div} \sigma(w), \psi) = -(\sigma(w), \epsilon(\psi)) + \langle l, \psi \rangle \text{ in } H^{-1}([0,T])$$

and thus for all $\psi \in H^1(\Omega_s)$. Now by Green's theorem applied to the left hand side, we obtain:

$$-(\sigma(w), \epsilon(\psi)) + \langle \sigma(w).\nu, \psi \rangle = -(\sigma(w), \epsilon(\psi)) + \langle l, \psi \rangle \text{ in } H^{-1}([0,T])$$

So, for all $\psi \in H^1(\Omega_s)$:

$$\langle \sigma(w).\nu, \psi \rangle = \langle l, \psi \rangle$$

However, for every $z \in H^{\frac{1}{2}}(\Gamma_s)$, there exists $\psi \in H^1(\Omega_s)$ such that $\psi|_{\Gamma_s} = z$. Hence, by the surjectivity of the trace operator we have for all $z \in H^{\frac{1}{2}}(\Gamma_s)$

$$\langle \sigma(w).\nu, z \rangle = \langle l, z \rangle \Rightarrow l = \sigma(w).\nu \in H^{-1}([0,T];H^{-\frac{1}{2}}(\Gamma_s))$$

However, since $l \in L_2([0,T];H^{-\frac{1}{2}}(\Gamma_s))$, $\sigma(w).\nu \in L_2([0,T];H^{-\frac{1}{2}}(\Gamma_s))$.

Therefore, the pair of limits w, w_t satisfy the equation (3.34).

Step 3: Passing through the limit in the Fluid Equation

- With the uniform bound on u_n in $L_2([0,T];V)$ given in (3.38), we can extract a subsequence u_n converging weakly in $L_2([0,T];V)$ to some u.
- Given the uniform bounds on u_n and u_{nt} in (3.37) and (3.47), we can use a compactness result (Aubin's Lemma), to conclude that there exists a subsequence u_{m_j} that is convergent strongly in $L_2([0,T];H)$ to some function $u \in L_2([0,T];H)$.

Now, we are ready to pass through the limit in (3.35). Let $v \in V$ be time independent and take (3.35) and integrate time to obtain:

$$(u_{n'}(t), v) + \int_{t_0}^{t} (\epsilon(u_{n'})(s), \epsilon(v))ds + \int_{t_0}^{t} (B_{n'}(u_{n'})(s), v)ds \tag{3.52}$$

$$= \langle u_{n'}(t_0), v \rangle + \int_{t_0}^{t} \langle \sigma(w_{n'})(s).\nu, v \rangle ds \tag{3.53}$$

Since $u_{n'}$ converges to u weakly in $L_2([0,T];V)$, $u_{n'}(t)$ converges in V weakly to $u(t)$ almost everywhere. Hence, since V is compactly embedded in H, $u_{n'}(t)$ converges strongly almost everywhere to $u(t)$ in H. Thus, taking the limit as $n \to \infty$ in (3.52), we have by strong convergence almost everywhere in H:

$$\lim_{n \to \infty} (u_{n'}(t), v) = (u(t), v) \ , \ \lim_{n \to \infty} (u_{m'}(t_0), v) = (u(t_0), v) \tag{3.54}$$

On the other hand, weak convergence of the sequence $u_{n'}$ in $L_2([0,T];V)$ yields:

$$\lim_{n\to\infty}\int_{t_0}^{t}(\epsilon(u_{n'})(s),\epsilon(v))ds=\int_{t_0}^{t}(\epsilon(u(s)),\epsilon(v))ds \tag{3.55}$$

Finally, weak convergence of $\sigma(w_n).\nu \to l = \sigma(w).\nu$ on $L_2([0,T];H^{-\frac{1}{2}}(\Gamma_s))$ by (3.48) yields:

$$\lim_{n\to\infty}\int_{t_0}^{t}\langle\sigma(w_{n'})(s,.).\nu,v\rangle ds=\lim_{n\to\infty}\int_{t_0}^{t}\langle\sigma(w)(s,.).\nu,v\rangle ds \tag{3.56}$$

Step 4: Passing through the limit in the nonlinear term

It remains to show

$$\int_{t_0}^{t}(B_n(u_n)-Bu,v)ds\to 0 \tag{3.57}$$

In what follows we shall use $|u|_r$ to indicate the H^r norm on Ω_f i.e.:

$|u|_r \equiv |u|_{r,\Omega_f} \equiv |u|_{H^r(\Omega_f)}$

Let $S_n=\{t\in(0,T);|u_n(t)|_1\geq n\}$ and denote by $m(S_n)$ the measure of S_n

Notice that $m(S_n)\leq\frac{E(0)}{n^2}$ since by (3.38) $E(0)\geq\int_{S_n}|u_n(s,.)|_1^2ds\geq m(S_n)n^2$.

Let us then estimate $\int_{t_0}^{t}(B_n(u_n)-Bu,v)ds$ on the complement of S_n (i.e. the time values s for which $B_nu(s,.)=Bu(s,.)$).

Estimate of $\int(B_n(u_n)-Bu,v)ds$ on $\overline{S_n}$

$$\begin{aligned}\int_{\overline{S_n}}(B_n(u_n)-Bu,v)ds&=\int_{\overline{S}}b(u_n,u_n,v)-b(u,u,v)ds\\&=\int_{\overline{S_n}}[b(u_n-u,u_n,v)+b(u,u_n-u,v)]ds\\&\leq\int_{\overline{S_n}}|b(u_n-u,u_n,v)|+|b(u,v,u_n-u)|ds\\&\leq\int_0^T|u_n-u|_{3/4}|v|_1[|u_n|_1+|u|_1]ds\\&\leq|v|_1\int_0^T|u_n-u|_0^{1/4}|u_n-u|_1^{3/4}ds[|u_n|_1+|u|_1]\\&\leq|v|_1(\int_0^T|u_n-u|_0^2)^{1/8}(\int_0^T|u_n-u|_1^2ds)^{3/8}[\int_0^T|u_n|_1^2ds+\int_0^T|u|_1^2]^{\frac{1}{2}}\end{aligned}$$

The first integral goes to zero by strong convergence of $u_n\to u\in L_2([0,T],H)$ while the second integral can be shown to be uniformly bounded by weak convergence of $u_n\to u\in L_2([0,T];V)$. Hence, by taking the limit the left hand side converges to zero.

Estimate of $\int(B_n(u_n)-Bu,v)ds$ on S

$$\begin{aligned}|\int_{S_n}(B_n(u_n)-Bu,v)ds|&=|\int_{S_n}[\frac{n^2}{|u_n|_V^2}b(u_n,u_n,v)-b(u,u,v)]ds\\\leq\int_{S_n}|b(u_n,u_n,v)|+|b(u,u,v)|ds&\leq|v|_1[\int_S|u_n|_0^{\frac{1}{2}}|u_n|_1^{3/2}ds+\int_{S_n}|u|_0^{\frac{1}{2}}|u|_1^{3/2}ds]\\&\leq|v|_1[(\int_{S_n}|u_n|_0^2ds)^{1/4}(\int_0^T|u_n|_1^2ds)^{3/4}+(\int_{S_n}|u|_0^2ds)^{1/4}(\int_0^T|u|_1^2ds)^{3/4}]\end{aligned}$$

$$\leq 2|v|_1[\frac{E^{5/4}(0)}{n^{1/2}}]$$

Where in the final estimate we used the uniform boundedness of u_n in $L_2([0,T];V)$ by (3.38) and in $L_\infty([0,T];H)$ by (3.37) respectively. Finally letting $n \to 0$, the right hand side goes to zero. Hence, $\int_{t_0}^t (B_n u_n)(s,.) - Bu(s,.), v)ds \to 0$, and the proof of wellposedness is complete when combining (3.54), (3.55), (3.56) and (3.57) which verifies that the limit u satisfies equation (3.1). This complete the proof of global existence of weak solutions. □

4. Uniqueness of Weak Solutions in 2 dimensions

THEOREM 4.1. *Let $\Omega \subset \mathbb{R}^2$. A weak solution to the system given in (1.1) as defined in (2.1) is unique.*

PROOF. Given initial data $[u_0, w_0, w_1] \in \mathcal{H}$, assume there exists two solutions of the variational form (2.1), $[u^{(1)}, w^{(1)}, w_t^{(1)}]$ and $[u^{(2)}, w^{(2)}, w_t^{(2)}]$.

Then both solutions belong to $L^2([0,T];V) \cap L_\infty([0,T];H) \times L_\infty([0,T];H^1(\Omega_s) \times L_2(\Omega_s))$. Let $z = u^{(2)} - u^{(1)}$, and $v = w^{(2)} - w^{(1)}$, then z, v satisfy the following system for all $\phi \in V$ and $\psi \in H^1(\Omega_s)$ a.e. $0 \leq t \leq T$:

$$(z_t, \phi) + (\epsilon(z), \epsilon(\phi)) + b(u^{(1)}, z, \phi) + b(z, u^{(2)}, \phi) + \langle \sigma(v).\nu, \phi \rangle = 0 \tag{4.1}$$

$$(v_{tt}, \psi) + (\sigma(v), \epsilon(\psi)) - \langle \sigma(v).\nu, \psi \rangle = 0 \tag{4.2}$$

$$v_t|_{\Gamma_s} = z|_{\Gamma_s} \tag{4.3}$$

$$z(0,.) = 0 \qquad \Omega_f \tag{4.4}$$

$$v(0,.) = 0 \quad \Omega_s, v_t(0,.) = 0 \quad \Omega_s \tag{4.5}$$

Now take $\phi = z$ and $\psi = v_t$ proceed to obtain the following estimates noting that $b(u, z, z) = 0$ from lemma 3.4 to obtain:

$$\frac{1}{2}\frac{d}{dt}|z|^2_{0,\Omega_f} + |\epsilon(z(t))|^2_{0,\Omega_f} + b(z(t), u^{(2)}(t), z(t)) + \langle \sigma(v)(t).\nu, z(t) \rangle = 0$$

$$\frac{1}{2}\frac{d}{dt}|v_t|^2_{0,\Omega_s} + \frac{1}{2}\frac{d}{dt}(\sigma(v(t)), \epsilon(v(t)))_{\Omega_s} - \langle \sigma(v)(t).\nu, v_t(t) \rangle \leq 0$$

Now , we again combine the two equations via cancelation of $\langle \sigma(v).\nu, v_t \rangle$ since $z|_{\Gamma_s} = v_t|_{\Gamma_s}$:

$$|\epsilon(z(t))|^2_{0,\Omega_f} + \frac{1}{2}\frac{d}{dt}[|z|^2_{0,\Omega_f} + |v_t|^2_{0,\Omega_s} + (\sigma(v), \epsilon(v))_{\Omega_s}] \leq -b(z(t), u^{(2)}(t), z(t)) \tag{4.6}$$

Now , we use the properties of the trilinear form $b(u, v, w)$ in dimension $n = 2$ from lemma 3.4(3) to estimate the trilinear term $b(z, u^{(2)}, z)$:

$$|b(z, u^{(2)}, z)| \leq c_1 |z|_{\frac{1}{2},\Omega_f} |u^{(2)}|_{1,\Omega_f} |z|_{\frac{1}{2},\Omega_f} + c_2 |z|_{\frac{1}{2},\Omega_f} |u^{(2)}|_{3/4,\Omega_f} |z|_{3/4,\Omega_f} \tag{4.7}$$

$$\leq c_1 |z|_{0,\Omega_f} |z|_{1,\Omega_f} |u^{(2)}|_{1,\Omega_f} + c_2 |z|^{3/4}_{0,\Omega_f} |z|^{5/4}_{1,\Omega_f} |u^{(2)}|^{3/4}_{1,\Omega_f} |u^{(2)}|^{1/4}_{0,\Omega_f}$$

Where the last estimate follows from interpolation between the spaces $H^{\frac{1}{2}}(\Omega_f)$ and $H^{3/4}(\Omega_f)$. Moreover, applying Young's inequality we have:

(4.8)
$$|b(z,u^{(2)},z)| \le 4c_1^2|z|_{0,\Omega_f}^2|u^{(2)}|_{1,\Omega_f}^2 + 1/2|z|_{1,\Omega_f}^2 + (4)^{3/5}c_2^{8/3}|z|_{0,\Omega_f}^2|u^{(2)}|_{1,\Omega_f}^2|u^{(2)}|_{0,\Omega_f}^{2/3}$$

Hence, after setting $y(t) = [|z(t)|_{0,\Omega_f}^2 + |v_t(t)|_{0,\Omega_s}^2 + (\sigma(v(t)),\epsilon(v(t)))_{\Omega_s}]$, $K_1 = 4c_1^2$ and $K_2 = (4)^{3/5}c_2^{8/3}$, then using (4.7), the estimate (4.6) becomes:

$$\frac{1}{2}|z(t)|_{1,\Omega_f}^2 + \frac{1}{2}\frac{d}{dt}y(t) \le |z(t)|_{0,\Omega_f}^2|u^{(2)}(t)|_{1,\Omega_f}^2[K_1 + K_2|u^{(2)}(t)|_{0,\Omega_f}^{2/3}]$$

Next, we drop the term $|z|_{1,\Omega_f}$ and integrate in time to obtain

$$y(t) \le y(0) + \int_0^t |z(s)|_{0,\Omega_f}^2|u^{(2)}(s)|_{1,\Omega_f}^2[K_1 + K_2|u^{(2)}(s)|_{0,\Omega_f}^{2/3}]ds$$

$$\le y(0) + \int_0^t y(s)|u^{(2)}(s)|_{1,\Omega_f}^2[K_1 + K_2|u^{(2)}(s)|_{0,\Omega_f}^{2/3}]$$

Finally, we apply Gronwall's inequality and use the boundedness of any weak solution $u^{(2)} \in L_2([0,T];V)$ and $\in L_\infty([0,T];H)$:

$$y(t) \le y(0)e^{[K_1+K_2|u^{(2)}|_{L_\infty([0,T];H)}^{2/3}]\int_0^t |u^{(2)}(s)|_{1,\Omega_f}^2 ds}$$

However, the initial condition $y(0) = |z(0,.)|_{0,\Omega_f}^2 + |v(0,.)_t|_{0,\Omega_s}^2 + (\epsilon(v(0,.)),\sigma(v(0,.)))_{\Omega_s}$ is identically 0 and the right hand side of the inequality is finite. Hence, $y(t) = 0$ a.e. t and thus $[z, v, v_t]$ is zero and this implies that the two assumed weak solutions must coincide, so uniqueness follows. □

5. Proof of the Trace Regularity for the Wave Equation in Theorem 3.3

Step 1: Decomposition By applying the principle of superposition we may split the problem into two subproblems. The first problem is driven by the initial conditions and zero boundary conditions, while the second problem is driven by $w_t = g$ on the boundary . More specifically, let Dg denotes Dirichlet map associated with $\operatorname{div}\sigma(w)$ and Dirichlet boundary conditions imposed on Γ_s. This is to say $\operatorname{div}\sigma[Dg] = 0,\ in\ \Omega_s$ and : $Dg|_{\Gamma_s} = g\ in\ \Gamma_s$. We recall that elliptic theory gives $D : H^{\frac{1}{2}}(\Gamma_s) \to H^1(\Omega_s)$ is bounded. We decompose w as $w = \hat{w} + w^*$ where both $\hat{w}$ and w^* satisfy the same problem (3.21) but the following boundary and initial conditions:

$$\hat{w}(0) = w_0 - D(w_0|_{\Gamma_s}) \in H_0^1(\Omega_s);\ \hat{w}_t(0) = w_1 \in L_2(\Omega_s)$$
$$\hat{w} = 0\ on\ \Gamma_s \tag{5.1}$$

$$w^*(0) = D(w_0|_{\Gamma_s}) \in H^1(\Omega);\ w_t^*(0) = 0$$
$$w_t^* = g,\ on\ \Gamma_s \tag{5.2}$$

Step 2: Analysis of $\hat{w}$ The regularity of $\hat{w}$ follows from the theory presented in [**6, 23**] and exploiting "hidden" regularity of the Dirichlet wave problem. More specifically, we have

$$\hat{w} \in C([0,T];H_0^1(\Omega_s)),\ \hat{w}_t \in C([0,T];L_2((\Omega_s))$$
$$\frac{\partial}{\partial\nu}\hat{w} \in L_2([0,T].L_2(\Gamma_s)) \tag{5.3}$$

and for $t \leq T$:

$$|\hat{w}(t)|_{1,\Omega_s} + |\hat{w}_t(t)|_{0,\Omega_s} + |\frac{\partial}{\partial \nu}\hat{w}|_{L_2(\Sigma_s)} \leq C_t[|w_0|_{1,\Omega_s} + |w_1|_{0,\Omega_s}] \tag{5.4}$$

Thus, it suffices to analyze the variable w^* only.

Step 3: analysis of w^* in the interior. For this, we shall apply Theorem 10.5.7.1 in **[18]** Since

$$w^*(0) = D(w_0|_{\Gamma_s}) \in H^1(\Omega_s), \;\; w_t^*(0) = 0$$

$$w^*|_{\Gamma_s} = \int_0^t g(s)ds + w_0|_{\Gamma_s} \in H^1([0,T]; L_2(\Gamma_s)) \cap C([0,T]; H^{\frac{1}{2}}(\Gamma_s))$$

With the compatibility condition

$$w^*(0)|_{\Gamma_s} = w_0|_{\Gamma_s} = w^*|_{\Gamma_s}(t=0) \tag{5.5}$$

we are in a position to apply Theorem 10.5.7.1 page 967 **[18]** to infer

$$w^* \in C([0,T]; H^1(\Omega_s)), w_t \in C([0,T]; L_2(\Omega_s)) \tag{5.6}$$

with the inequality:

$$|w^*(t)|_{1,\Omega_s} + |w^*(t)_t|_{0,\Omega_s} \leq C[|w_0|_{1,\Omega_s} + |g|_{L_2([0,T];H^{\frac{1}{2}}(\Gamma_s))}] \tag{5.7}$$

Step 4: boundary regularity of w^* Thus, it suffices to establish the boundary regularity of $\frac{\partial}{\partial \nu} w^*$. This amounts to considering the following problem:

$$w_{tt}^* = \operatorname{div} \sigma(w^*), \;\; in \; \Omega_s \times [0,T]$$

$$w^*(0) = D(w_0|_{\Gamma_s}) \in H^1(\Omega_s);$$

$$w_t^*(0) = 0 \tag{5.8}$$

$$w_t^*|_{\Gamma_s} = g \;\in L_2([0,T]; H^{\frac{1}{2}}(\Gamma_s)) \tag{5.9}$$

The key part of the proof is the following trace result:

PROPOSITION 5.1.

$$\frac{\partial}{\partial \nu} w^* \in L_2([0,T], H^{-\frac{1}{2}}(\Gamma_s)) \tag{5.10}$$

with the estimate:

$$|\frac{\partial}{\partial \nu} w^*|_{L_2([0,T];H^{-\frac{1}{2}}(\Gamma_s))} \leq C(|w^*(0)|_{1,\Omega_s} + |g|_{L_2([0,T];H^{\frac{1}{2}}(\Omega_s))})$$
$$\leq C(|w_0|_{1,\Omega_s} + |g|_{L_2([0,T];H^{\frac{1}{2}}(\Omega_s))})$$

Assuming for a moment the validity of proposition 5.1 , the result stated in Theorem 3.3 follows directly by combining (5.4), (5.7) and Proposition 5.1. Thus, in order to complete the proof of Theorem 3.3 we need to establish the validity of proposition 5.1. This is done below.

Step 5: proof of Proposition 5.1

In what follows we shall write w instead of w^*, and we consider the problem in Theorem 3.3 with the elastic operator $\operatorname{div} \sigma$ replaced by the Laplacian. The same result in (5.1) will then immediately follow for the system of elasticity in Theorem 3.3, see **[17]**. We first proceed to flatten the boundary via a Melrose Sjostrand change of coordinates. So we let $\{U_j, T_j\}$ be a chart of local coordinates with U_j an open cover of Ω_s and T_j the corresponding map into the half plane so that every point $\xi_1, \xi_2 \in \Omega_s$ is given by a point in the half plane $x > 0$. In addition we let ψ_j be the partition of unity subordinate to the open cover U_j so that ψ_j has compact support in U_j. Therefore, we first consider a reformulation of the problem for those $\psi_j w|_{U_j}$ over collar domains $M_j = U_j \bigcap \Omega$ which contain parts of the boundary Γ_s, and which vanish at $\partial U_j \bigcap \Omega_s$. With the appropriate change of coordinates, we obtain a new problem over the domain $\Omega_c = \{0 \leq x < 1 : |y| < 1\}$ corresponding to the local problem over the collar domain M and such that $\psi_j w|_{U_j} T_j^{-1}$ vanishes on $x = 1$ and $|y| = 1$ part of the boundary and such that the boundary condition on $x = 0$ corresponds to that on $\Gamma_s \bigcap M_j$. Since Ω_s is bounded and hence can be covered by finitely many sets, it suffices to prove the trace regularity result locally for the problem on Ω_c with $\Gamma_c = \{0\} \times [-1, 1]$. With the change of coordinates and via partition of unity, the new local problem we obtain for w over Ω_c is the following problem $Pw = 0$:

$$Pw = w_{tt} - \tilde{\Delta} w = 0 \ , \ \Omega_c \times [0, T] \tag{5.11}$$

$$w_t = g \ , \ \Gamma_c \times [0, T] \tag{5.12}$$

with $\tilde{\Delta} = D_x^2 + R(x, y, D_y)$ and $R(x, y, D_y) = \rho(x, y) D_y^2 +$ L.O.T. in D_y, and where $\rho(x, y)$ is a smooth function in x and y.

5.1. Time Localization. Let $\phi(t) \in C_0^\infty(-\infty, \infty)$ be a cut off function:

$$\phi(t) \equiv 1 \ , \ t \in [\epsilon, T - \epsilon] \tag{5.13}$$

$$\phi(t) \equiv 0 \ , \ t \leq 0, \ \phi(t) \equiv 0 \ , \ t \geq T \tag{5.14}$$

Now, define a new variable w_c defined on $\Omega_c \times (-\infty, \infty)$:

$$w_c(t, x, y) = \phi(t) w(t, x, y) \tag{5.15}$$

Therefore:

$$w_c(0, x, y) = 0, \ w_{c,t}(0, x, y) = 0 \tag{5.16}$$

LEMMA 5.1. *In term of the new variable w_c the original problem becomes over $Q_{c,\infty} = (-\infty, \infty) \times \Omega_c$:*

$$w_{c,tt} - \tilde{w}_c = [P, \phi] w = \phi_{tt} w + 2\phi_t w_t \tag{5.17}$$

$$w_{c,t}|_{x=0} = \phi g + \phi_t w|_{x=0} \tag{5.18}$$

where the commutator $[P, \phi]$ is given by:

$$[P, w] w = \phi_{tt} w + 2\phi_t w_t \tag{5.19}$$

PROOF. Direct verification □

5.2. Dual Space Localization. We will microlocalize the problem by moving to the Fourier space in both time and the tangential space variables with $t \to i\sigma$ and $y \to i\eta$. By symmetry we will consider the quarter Fourier space $R_{+,\sigma,\eta} = \{\sigma > 0, \eta > 0\}$ of $\mathbb{R}$ of σ, η. We divide the space in three cones $R_1 \bigcup R_{tr} \bigcup R_2$ defined as follows:

$$R_1 = \{(\sigma, \eta) \in \mathbb{R}_{+,\sigma\eta} : \sigma > c_1 \eta\} \tag{5.20}$$

$$R_{tr} = \{(\sigma, \eta) \in \mathbb{R}_{+,\sigma\eta} : c_2 \eta \leq \sigma \leq c_1 \eta\} \tag{5.21}$$

$$R_2 = \{(\sigma, \eta) \in \mathbb{R}^2_{+,\sigma\eta} : 0 \leq \sigma \leq c_2 \eta\} \tag{5.22}$$

We choose $c_2 = \frac{\sqrt{3\alpha}}{2}$ where α is the ellipticity constant for $\rho(x, y) D_y^2$.

Let $\chi(x,y;\sigma,\eta) \in S^0(\mathbb{R}^3_{tyx})$ be a homogeneous symbol of localization of order zero (i.e. C_0^∞ homogeneous function of order zero in both σ,η) such that:

$$\chi(\sigma,\eta) \equiv 1 \ in \ R_1 \tag{5.23}$$

$$supp\ \chi \subset R_1 \bigcup R_{tr}$$

$$1-\chi(\sigma,\eta) \equiv 1 \ in \ R_2 \tag{5.24}$$

$$supp\ (1-\chi) \subset R_2 \bigcup R_{tr}$$

Let $X \in OPS^0(\mathbb{R}^3_{tyx})$ be the corresponding pseudodifferential operator to the symbol χ. In other words $X : H^s(Q_{c,\infty}) \to H^s(Q_{c,\infty})$ continuously.

5.3. Localized Xw_c Problem. We write:

$$w_c = Xw_c + (1-X)w_c = w_1 + w_2 \tag{5.25}$$

Since $w_c \in H^1(Q_{c,\infty})$ we have that:

$$w_1, w_2 \in C(\mathbb{R}_t; H^1(\Omega_c)) \bigcap C^1(\mathbb{R}_t; L_2(\Omega_c)) \tag{5.26}$$

We define the commutator for P, X where P represents the problem as $[P,X]v = PXv - XPv$ so that $[P,X] \in OPS^1 : H^s(Q_{s,\infty}) \to H^{s-1}(Q_{c,\infty})$ continuously.

LEMMA 5.2. *The new variable $w_1 = Xw_c$ solves the following localized mixed problem:*

$$w_{1,tt} - \bar{\Delta} w_1 = f \ in \ Q_{c,\infty} = (-\infty,\infty) \times \Omega_c \tag{5.27}$$

$$w_{1,t}(t,0,y) = q(t,y) \tag{5.28}$$

where

$$f = [P,X]w_c + X\phi_{tt}w + 2X\phi_t w_t \in L_2(Q_{c,\infty}) \tag{5.29}$$

$$q = X\phi g + X\phi_t w + X_t w_c \in L_2(\Sigma_{c,\infty}) \tag{5.30}$$

PROOF. Direct verification. □

LEMMA 5.3. *With reference to the cone R_1, the localized problem for w_1 (5.27), (5.28), the following estimate holds:*

$$\int_{\Sigma_{c,\infty}} |w_{1,y}(t,0,y)|^2 d\Sigma_{c,\infty} \le \int_{\Sigma_{c,\infty}} |w_{1,t}(t,0,y)|^2 d\Sigma_{c,\infty} \le \infty \tag{5.31}$$

PROOF. Let $\hat{w}_1(\sigma,0,\eta)$ be the Fourier transform of $w_1(t,0,y)$, so $\hat{w}_1(\sigma,0,\eta) = \chi(\sigma,\eta)\hat{w}_c(\sigma,\eta)$ and supp $\hat{w}_1 \subset R_1 \bigcap R_{tr}$ since $w_1 = Xw_c$. Notice that $\sigma > c_2\eta$ in $R_1 \bigcap R_{tr}$. Now, we estimate using Plancheral theorem:

$$\int_{\Sigma_{c,\infty}} |w_{1,y}(t,0,y)|^2 d\Sigma_{c,\infty} = \int_{R_1 \bigcap R_{tr}} \eta^2 |\hat{w}_1(\sigma,0,\eta)|^2 d\sigma d\eta$$

$$\le \frac{1}{(c_2)^2} \int \sigma^2 |\hat{w}_1(\sigma,0,\eta)|^2 d\sigma d\eta = \frac{1}{(c_2)^2} \int_{\Sigma_{c,\infty}} |w_{1,t}(t,0,y)|^2 d\Sigma_{c,\infty}$$

$$= \frac{1}{(c_2)^2} \int_{\Sigma_{c,\infty}} |g|^2 d\Sigma_{c,\infty} \le \infty$$

□

THEOREM 5.4. *With reference problem (5.27), (5.28), w_1 satisfies the following trace regularity result:*

$$w_{1,x}(t,0,y) = \frac{\partial}{\partial \nu} w_1|_{\Gamma_c} \in L_2(\Sigma_{c,\infty}) \tag{5.32}$$

and

$$|w_{1,x}(t,0,y)|_{L_2(\Sigma_{c,\infty})} \leq C|w_1|_{H^1(\Sigma_{c,\infty})} \tag{5.33}$$

PROOF. As a corollary of lemma 5.3, $w_{1,y}(t,0,y), w_{1,t}(t,0,y) \in L_2(\Sigma_{c,\infty}) = L_2(\mathbb{R}_t; L_2(\Gamma_c))$. Therefore, $w_1(t,0,y) = w_1|_{\Gamma_c} \in H^1(\Sigma_{c,\infty})$, and thus since w_1 satisfies the wave equation (5.27), (5.28) problem, with $f \in L_2(Q_{c,\infty})$ and $w_1|_{\Gamma_c} \in H^1(\Sigma_{c,\infty})$, we appeal to the trace regularity result in **[23]** implying that $w_{1,x}(t,0,y) = \frac{\partial}{\partial \nu} w_1|_{\Gamma_c} \in L_2(\Sigma_{c,\infty})$ with continuous dependence on initial data and boundary condition. □

5.4. Localized w_2 Problem.

LEMMA 5.5. *The new variable $w_2 = (1-X)w_c$ solves the following localized mixed problem:*

$$w_{2,tt} - \tilde{\Delta} w_2 = h \text{ in } Q_{c,\infty} = (-\infty,\infty) \times \Omega_c \tag{5.34}$$

$$w_{2,t}(t,0,y) = s(t,y) \tag{5.35}$$

where

$$h = [P, 1-X]w_c + (1-X)\phi_{tt} w + 2(1-X)\phi_t w_t \in C(\mathbb{R}^t; L^2(\Omega_c)) \tag{5.36}$$

$$s = (1-X)\phi g + (1-X)\phi_t w \in L_2(\mathbb{R}^t; H^{\frac{1}{2}}(\Gamma_c)) \tag{5.37}$$

PROOF. Direct verification. □

LEMMA 5.6. *The operator P of problem (5.34), (5.35) defined as $Pw_2 \equiv w_{2,tt} - \tilde{\Delta} w_2$ is an elliptic operator on the image space of $H_0^1(Q_{c,\infty})$ under $1-X$.*

PROOF.

$$P(x,y;D_t,D_x,D_y) = -D_t^2 + \rho(x,y)D_y^2 + D_x^2$$

We consider the characteristic polynomial of P and denote it by p:

$$p(x,y;\sigma,\eta,\zeta) = -\sigma^2 + \rho(x,y)\eta^2 + \zeta^2$$

We now estimate from below using the condition $\sigma < \frac{\sqrt{3}}{2}\sqrt{\alpha}\eta$ in $R_2 \bigcap R_{tr}$:

$$\begin{aligned} p(x,y;\sigma,\eta,\zeta) &= -\sigma^2 + \rho(x,y)\eta^2 + \zeta^2 \\ &\geq -3/4\alpha\eta^2 + \alpha\eta^2 + \zeta^2 \\ &\geq 1/4\alpha\eta^2 + \zeta^2 \\ &\geq 1/8\alpha\eta^2 + 1/6\sigma^2 + \zeta^2 \\ &\geq C[\eta^2 + \sigma^2 + \zeta^2] \end{aligned}$$

where C is the minimum of the $\{1/6, 1/8\alpha\}$. Ellipticity of P in all the variables x, y, t then follows from the above estimate. □

THEOREM 5.7. *With reference problem (5.34), (5.35), w_2 satisfies the following trace regularity result:*

$$w_{2,x}(t,0,y) = \frac{\partial}{\partial\nu} w_2|_{\Gamma_c} \in C(\mathbb{R}^t; H^{-\frac{1}{2}}(\Gamma_c)) \tag{5.38}$$

and

$$|w_{2,x}(t,0,y)|_{C(\mathbb{R}^t;H^{-\frac{1}{2}}(\Gamma_c))} \leq C(|w_2|_{C(\mathbb{R}^t;H^{\frac{1}{2}}(\Gamma_c))} + |h|_{C(\mathbb{R}^t;L^2(\Omega_c))}) \tag{5.39}$$

PROOF. For every $t \in (-\infty, \infty)$, w_2 satisfies the elliptic problem:

$$Pw_2 = h \in C(\mathbb{R}^t; L_2(\Omega_c))$$

$$w_2|_{\Gamma_c} = w_2(t,0,y) = \int_0^t w_{2,t}(\tau,0,y) \in C(\mathbb{R}^t; H^{\frac{1}{2}}(\Gamma_c))$$

Therefore, we can invoke elliptic theory regularity results:

$$|w_{2,x}(t,0,y)|_{C(\mathbb{R}^t;H^{-\frac{1}{2}}(\Gamma_c))} \leq C(|w_2|_{C(\mathbb{R}^t;H^{\frac{1}{2}}(\Gamma_c))} + |h|_{C(\mathbb{R}^t;L^2(\Omega_c))})$$

□

5.5. Final Result: Proof of Proposition.

PROOF. By Theorems 5.4, 5.7:

$$\frac{\partial}{\partial\nu} w_c = \frac{\partial}{\partial\nu} w_1 + \frac{\partial}{\partial\nu} w_2 \in L_2(\mathbb{R}^t; H^{-\frac{1}{2}})(\Gamma_c)$$

Now we estimate $|\frac{\partial}{\partial\nu} w|_{L_2([\epsilon,T-\epsilon];H^{-\frac{1}{2}}(\Gamma_c))}$:

$$\begin{aligned}
|\frac{\partial}{\partial\nu} w|_{L_2([\epsilon,T-\epsilon];H^{-\frac{1}{2}}(\Gamma_c))} &\leq |\frac{\partial}{\partial\nu} w_c|_{L_2(\mathbb{R}^t;H^{-\frac{1}{2}}(\Gamma_c))} \\
&\leq |\frac{\partial}{\partial\nu} w_1|_{L_2(\mathbb{R}^t;H^{-\frac{1}{2}}(\Gamma_c))} + |\frac{\partial}{\partial\nu} w_2|_{L_2(\mathbb{R}^t;H^{-\frac{1}{2}}(\Gamma_c))} \\
\leq C(|w_2|_{C(\mathbb{R}^t;H^{\frac{1}{2}}(\Gamma_c))} &+ |h|_{C(\mathbb{R}^t;L^2(\Omega_c))} + |f|_{C(\mathbb{R}^t;L^2(\Omega_c))} + |w_1|_{H^1(\Sigma_{c,\infty})}) \\
&\leq K(|w_0|_{0,\Omega_s} + |w_1|_{1,\Omega_s} + |g|_{L_2([0,T];H^{\frac{1}{2}}(\Gamma_c))})
\end{aligned}$$

The results is then established for the local problem on Ω_c and hence for all the domain Ω_s. □

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