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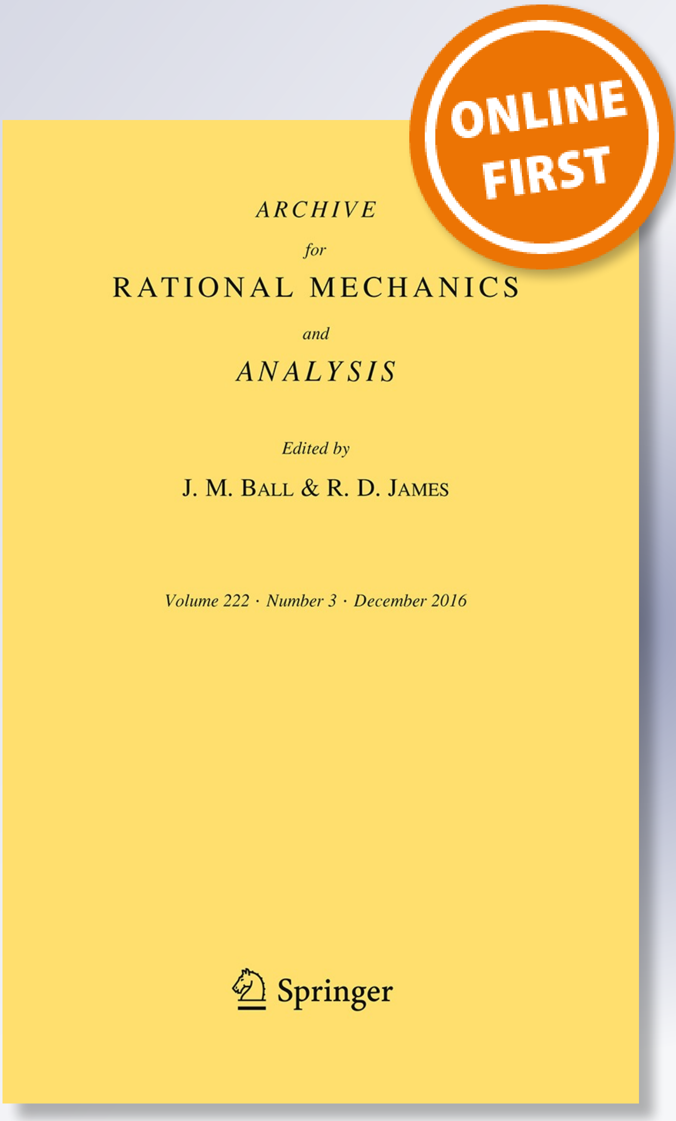

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Frequency Localized Regularity Criteria for the 3D Navier–Stokes Equations

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Abstract

Two regularity criteria are established to highlight which Littlewood–Paley frequencies play an essential role in possible singularity formation in a Leray–Hopf weak solution to the Navier–Stokes equations in three spatial dimensions. One of these is a frequency localized refinement of known Ladyzhenskaya–Prodi–Serrin-type regularity criteria restricted to a finite window of frequencies, the lower bound of which diverges to $+\infty$ as t approaches an initial singular time.

1. Introduction

The Navier–Stokes equations governing the evolution of a viscous, incompressible flow's velocity field u in $\mathbb{R}^3 \times (0, T)$ read

$$\begin{aligned} &\partial_t u + u \cdot \nabla u = -\nabla p + \nu \Delta u + f && \text{in } \mathbb{R}^3 \times (0, T) \\ &\nabla \cdot u = 0 && \text{in } \mathbb{R}^3 \times (0, T), \end{aligned} \tag{3D NSE}$$

where ν is the viscosity coefficient, p is the pressure, and f is the forcing. For convenience we take f to be zero and set $\nu = 1$. The flow evolves from an initial vector field u_0 taken in an appropriate function space.

The regularity of Leray–Hopf weak solutions (i.e. distributional solutions for $u_0 \in L^2$ that satisfy the global energy inequality and belong to $L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$ for any $T > 0$) remains an open problem. The best results available rely on critical quantities being finite, that is quantities which are invariant given the natural scaling associated with the Navier–Stokes equations. In this note we provide several regularity criteria which highlight the essential role of high frequencies in a possibly singular Leray–Hopf weak solution.

Frequencies are interpreted in the Littlewood–Paley sense. Let $\lambda_j = 2^j$ for $j \in \mathbb{Z}$ be measured in inverse length scales and let B_r denote the ball of radius r centered at the origin. Fix a non-negative, radial cut-off function $\chi \in C_0^\infty(B_1)$ so that

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$\chi(\xi) = 1$ for all $\xi \in B_{1/2}$. Let $\phi(\xi) = \chi(\lambda_1^{-1}\xi) - \chi(\xi)$ and $\phi_j(\xi) = \phi(\lambda_j^{-1})(\xi)$. Suppose that u is a vector field of tempered distributions and let $\Delta_j u = \mathcal{F}^{-1}\phi_j * u$ for $j \in \mathbb{N}$ and $\Delta_{-1} = \mathcal{F}^{-1}\chi * u$. Then, u can be written as

$$u = \sum_{j \geq -1} \Delta_j u.$$

If $\mathcal{F}^{-1}\phi_j * u \to 0$ as $j \to -\infty$ in the space of tempered distributions, then for $j \in \mathbb{Z}$ we define $\dot{\Delta}_j u = \mathcal{F}^{-1}\phi_j * u$ and have

$$u = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u.$$

For $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$ the homogeneous Besov spaces include tempered distributions modulo polynomials for which the norm

$$\|u\|_{\dot{B}^s_{p,q}} := \begin{cases} \left(\sum_{j \in \mathbb{Z}} \left(\lambda_j^s \|\dot{\Delta}_j u\|_{L^p(\mathbb{R}^n)} \right)^q \right)^{1/q} & \text{if } q < \infty \\ \sup_{j \in \mathbb{Z}} \lambda_j^s \|\dot{\Delta}_j u\|_{L^p(\mathbb{R}^n)} & \text{if } q = \infty \end{cases},$$

is finite. See [2] for more details.

Given a Leray–Hopf weak solution u that belongs to $C(0, T; \dot{B}^{-\epsilon}_{\infty,\infty})$ for some ϵ in $(0, 1)$, we define the following upper and lower endpoint frequencies: for t in $(0, T)$ let

$$J_{\text{high}}(t) = \log_2 \left[c_1 \|u(t)\|_{\dot{B}^{-\epsilon}_{\infty,\infty}}^{1/(1-\epsilon)} \right], \tag{1}$$

and

$$J_{\text{low}}(t) = \log_2 \left[\left(c_2 \frac{\|u(t)\|_{\dot{B}^{-\epsilon}_{\infty,\infty}}}{\|u\|_{L^\infty(0,T;L^2)}} \right)^{2/(3-2\epsilon)} \right], \tag{2}$$

where c_1 and c_2 are universal constants (their values will become clear in Section 2). Our first regularity criterion shows J_{low} and J_{high} determine the Littlewood–Paley frequencies which, if well behaved at a finite number of times prior to a possible blow-up time, prevent singularity formation.

Theorem 1. *Fix* $\epsilon \in (0, 1)$ *and* $T > 0$, *and assume that* $u \in C(0, T; \dot{B}^{-\epsilon}_{\infty,\infty})$ *is a Leray–Hopf weak solution to 3D NSE on* $[0, T]$. *If there exists* $t_0 \in (0, T)$ *such that*

$$\sup_{J_{\text{low}}(t_0) \leq j \leq J_{\text{high}}(t_0)} \lambda_j^{-\epsilon} \|\dot{\Delta}_j u(t_i)\|_{L^\infty} \leq \|u(t_0)\|_{\dot{B}^{-\epsilon}_{\infty,\infty}}, \tag{3}$$

where $\{t_i\}_{i=1}^k \subset (t_0, T)$ *is a finite collection of k times satisfying*

$$t_{i+1} - t_i > \left(\frac{c_3}{\|u(t_0)\|_{\dot{B}^{-\epsilon}_{\infty,\infty}}} \right)^{2/(1-\epsilon)} \qquad (i = 0, \ldots, k-1),$$

and

$$T - t_k < \left(\frac{2c_3}{\|u(t_0)\|_{\dot{B}^{-\epsilon}_{\infty,\infty}}} \right)^{2/(1-\epsilon)}$$

for a universal constant c_3*, then* u *can be smoothly extended beyond time* T.

The novelty here is that the solution remains finite provided only a finite range of frequencies remain subdued at a finite number of uniformly spaced times. If u is not in the energy class then a partial result can be formulated since J_{high} does not depend on $\|u\|_{L^\infty(0,T;L^2)}$. In particular, we just need to replace (3) with

$$\sup_{j \le J_{\text{high}}(t)} \lambda_j^{-\epsilon} \|\dot{\Delta}_j u(t_i)\|_{L^\infty} \le \|u(t)\|_{\dot{B}^{-\epsilon}_{\infty,\infty}},$$

and assume u is the mild solution for $u_0 \in \dot{B}^{-\epsilon}_{\infty,\infty}$ which is a strong solution on $[0, T)$ (note that a local-in-time existence theory for mild solution is available in $\dot{B}^{-\epsilon}_{\infty,\infty}$).

Our second result is a refinement of a well known class of regularity criteria (see, e.g., [7]): if u is a Leray–Hopf weak solution to 3D NSE on $\mathbb{R}^3 \times [0, T]$ satisfying

$$\int_0^T \|u\|_{L^p}^q \, \mathrm{d}t < \infty,$$

for pairs (p, q) where $3 \le p \le \infty$, $2 \le q \le \infty$, and

$$\frac{2}{q} + \frac{3}{p} = 1,$$

then u is smooth. This is the Ladyzhenskaya–Prodi–Serrin class for non-endpoint values of (p, q). The case $p = \infty$ is the Beale–Kato–Majda regularity criteria. The case $p = 3$ was only (relatively) recently proven in [5]. Similar criteria can be formulated for a variety of spaces larger than L^p when $p > 3$. For example, Cheskidov and Shvydkoy give the following Ladyzhenskaya–Prodi–Serrin-type regularity criteria in Besov spaces (see [3]): if u is a Leray–Hopf solution and $u \in L^{2/(1-\epsilon)}(0, T; \dot{B}^{-\epsilon}_{\infty,\infty})$, then u is regular on $(0, T]$. A regularity criterion for weakly time integrable Besov norms in critical classes appears in [1]. In the endpoint case when $\epsilon = -1$, smallness is needed either over all frequencies (see [3]) or over high frequencies provided a Beale–Kato–Majda-type bound holds for the projection onto low frequencies (see [4]). Our result is essentially a refinement of the non-endpoint regularity criteria given in [3].

Theorem 2. *Fix* $\epsilon \in (0, 1)$ *and* $T > 0$*, and assume that* $u \in C(0, T; \dot{B}^{-\epsilon}_{\infty,\infty})$ *is a Leray–Hopf weak solution to 3D NSE on* $[0, T]$*. If*

$$\int_0^T \left(\sup_{J_{\text{low}}(t) \le j \le J_{\text{high}}(t)} \lambda_j^{-\epsilon} \|\dot{\Delta}_j u(t)\|_\infty \right)^{2/(1-\epsilon)} \mathrm{d}t < \infty,$$

then u *is regular on* $(0, T]$.

Clearly J_{high} blows up more rapidly than J_{low} as $t \to T^-$ and therefore an increasing number of frequencies are relevant as we approach the possible blow-up time. It is unlikely that this can be improved for weak solutions in supercritical classes like Leray–Hopf solutions. On one hand, the upper cutoff is available because of local well-posedness for the subcritical quantity $\|u(t)\|_{\dot{B}^{-\epsilon}_{\infty,\infty}}$ which suppresses high frequencies at times close to and after t. On the other hand, the supercritical quantity $\|u\|_{L^\infty(0,T;L^2)}$ plays a crucial role in suppressing low frequencies. Any supercritical quantity is sufficient; for example, if we replace $L^\infty L^2$ with $L^\infty L^p$ for some $2 < p < 3$, then the lower cutoff function is

$$J_{\text{low}}(t) = \log_2 \left[\left(\frac{\|u(t)\|_{\dot{B}^{-\epsilon}_{\infty,\infty}}}{c\|u\|_{L^\infty(0,T;L^p)}} \right)^{p/(3-p\epsilon)} \right].$$

Note that $p/(3-p\epsilon) = 1/(1-\epsilon)$ only when $p = 3$, i.e. the exponents in the cutoffs will match only when we reach a critical class $L^\infty(0, T; L^3)$.

2. Technical Lemmas

The local existence of strong solutions for data in the subcritical space $\dot{B}^{-\epsilon}_{\infty,\infty}$ is known, see [7]. Results in spaces close to $\dot{B}^{-\epsilon}_{\infty,\infty}$ are given in [6,9]. Indeed, the proof of [6, Theorem 1] can be modified to show that if $a \in \dot{B}^{-\epsilon}_{\infty,\infty}$, then the Navier–Stokes equations have a unique strong solution u which persists at least until time

$$T_* = \left(\frac{c_0}{\|a\|_{\dot{B}^{-\epsilon}_{\infty,\infty}}} \right)^{2/(1-\epsilon)}, \tag{4}$$

for a universal constant c_0. Moreover we have

$$\|u(t)\|_{\dot{B}^{-\epsilon}_{\infty,\infty}} \le c_0 \|a\|_{\dot{B}^{-\epsilon}_{\infty,\infty}}, \tag{5}$$

and

$$t^{1/2} \|\nabla u(t)\|_{\dot{B}^{-\epsilon}_{\infty,\infty}} \le c_0 \|a\|_{\dot{B}^{-\epsilon}_{\infty,\infty}}, \tag{6}$$

for any $t \in (0, T_*)$ (the value of c_0 changes from line to line but always represents a universal constant). Since the proof of this is nearly identical to the proof of [6, Theorem 1] it is omitted. Note that by [7, Proposition 3.2], the left hand side of (6) can be replaced by $t^{1/2}\|u\|_{\dot{B}^{1-\epsilon}_{\infty,\infty}}$.

Given a solution u and a time t so that $u(t) \in \dot{B}^{-\epsilon}_{\infty,\infty}$, let $t' = t + T_*/2$ and $t'' = t + T_*$ where T_* is as in (4) with $a = u(t)$. We now state and prove several (short) technical lemmas.

Lemma 3. *Fix* $\epsilon \in [0, 3/2)$ *and* $T > 0$*. If* u *is a Leray–Hopf weak solution to 3D NSE on* $[0, T]$ *and* $u(t) \in \dot{B}^{-\epsilon}_{\infty,\infty}$ *for some* $t \in [0, T]$*, then for any* $M > 0$ *we have*

$$\lambda_j^{-\epsilon} \|\dot{\Delta}_j u(t)\|_\infty \le M,$$

provided

$$j \le \log_2 \left[\left(c \frac{M}{\|u\|_{L^\infty(0,T;L^2)}} \right)^{2/(3-2\epsilon)} \right]$$

for a suitable universal constant c*.*

Proof. Assume u is a Leray–Hopf weak solution on $[0, T]$ and $t \in [0, T]$ such that $\|u(t)\|_{\dot{B}^{-\epsilon}_{\infty,\infty}} < \infty$. By Bernstein's inequalities we have

$$\|\dot{\Delta}_j u(t)\|_\infty \le \lambda_j^{3/2} \|\dot{\Delta}_j u(t)\|_2.$$

Since $u \in L^\infty(0, T; L^2) = L^\infty(0, T; \dot{B}^0_{2,2})$, for any $j \in \mathbb{Z}$,

$$\lambda_j^{-\epsilon} \|\dot{\Delta}_j u\|_\infty \le c\lambda_j^{3/2-\epsilon} \|u\|_{L^\infty(0,T;L^2)}.$$

Let

$$J(t) = \log_2 \left[\left(\frac{M}{c\|u\|_{L^\infty(0,T;L^2)}} \right)^{2/(3-2\epsilon)} \right];$$

then

$$\sup_{j \le J} \lambda_j^{-\epsilon} \|\dot{\Delta}_j u\|_\infty \le M.$$

Lemma 4. *Fix* $\epsilon \in (0, 1)$ *and* $T > 0$*, and assume* u *is a Leray–Hopf weak solution to 3D NSE on* $[0, T]$ *belonging to* $C(0, T; \dot{B}^{-\epsilon}_{\infty,\infty})$*. Then, for any* $t_1 \in (0, T)$ *and all* $t \in [t_1', t_1'']$ *we have*

$$\sup_{\{j \in \mathbb{Z}: j \le J_{\text{low}} \text{ or } j \ge J_{\text{high}}\}} \|\dot{\Delta}_j u(t)\|_{L^\infty} \le \frac{1}{2} \|u(t_1)\|_{\dot{B}^{-\epsilon}_{\infty,\infty}},$$

where J_{high} *and* J_{low} *are defined by* (1) *and* (2)*.*

Proof. Using subcritical local well-posedness in $\dot{B}^{-\epsilon}_{\infty,\infty}$ at t_1 we have that there exists a mild/strong solution v defined on $[t_1, t_1'']$. By (6) we have

$$(t - t_1)^{1/2} \|v(t)\|_{\dot{B}^{1-\epsilon}_{\infty,\infty}} \le c_0 \|v(t_1)\|_{\dot{B}^{-\epsilon}_{\infty,\infty}}$$

for all $t \in (t_1, t_1'')$. Since $v(t_1) = u(t_1) \in L^2$ and since the strong solution v is smooth, integration by parts verifies that v is also a Leray–Hopf weak solution to

3D NSE. The weak-strong uniqueness result of [8] then guarantees that $u = v$ on $[t_1, t_1'']$. Thus, for any $t \in [t_1', t_1'']$,

$$\lambda_j^{-\epsilon} \|\dot{\Delta}_j u(t)\|_\infty \leq c\lambda_j^{-1} \|u(t_1)\|_{\dot{B}^{-\epsilon}_{\infty,\infty}}^{1/(1-\epsilon)+1}$$

for all $j \in \mathbb{Z}$. By (1) we conclude that

$$\sup_{j \geq J_{\text{high}}} \lambda_j^{-\epsilon} \|\dot{\Delta}_j u(t)\|_\infty \leq \frac{1}{2} \|u(t_1)\|_{\dot{B}^{-\epsilon}_{\infty,\infty}}.$$

The low modes are eliminated using Lemma 3 with $M = \|u(t_1)\|_{\dot{B}^{-\epsilon}_{\infty,\infty}}/2$. □

Definition 5. We say that t is an *escape time* if there exists some $M > 0$ such that $t = \sup\{s \in (0, T) : \|u(s)\|_{\dot{B}^{-\epsilon}_{\infty,\infty}} < M\}$.

Lemma 6. *Fix* $\epsilon \in (0, 1)$ *and* $T > 1$, *and assume* u *is a Leray–Hopf weak solution to 3D NSE on* $[0, T]$ *belonging to* $C(0, T; \dot{B}^{-\epsilon}_{\infty,\infty})$. *Let* $\mathcal{E}$ *denote the collection of escape times in* $(0, T)$ *and let* $I = \cup_{t \in \mathcal{E}} (t', t'')$. *Then*

$$\int_0^T \|u(t)\|_{\dot{B}^{-\epsilon}_{\infty,\infty}}^{2/(1-\epsilon)} \, \mathrm{d}t = \infty, \tag{7}$$

if and only if

$$\int_I \|u(t)\|_{\dot{B}^{-\epsilon}_{\infty,\infty}}^{2/(1-\epsilon)} \, \mathrm{d}t = \infty. \tag{8}$$

Proof. It is obvious that (8) implies (7).

Assume (7). Let $\{t_k\}_{k \in \mathbb{N}} \subset (0, T)$ be an increasing sequence of escape times which converge to T at $k \to \infty$. Clearly $\|u(t_k)\|_{\dot{B}^{-\epsilon}_{\infty,\infty}}$ blows up as $k \to \infty$. Since $u \in C(0, T; \dot{B}^{-\epsilon}_{\infty,\infty})$, $\|u(t_{k_1})\|_{\dot{B}^{-\epsilon}_{\infty,\infty}} < \|u(t_{k_2})\|_{\dot{B}^{-\epsilon}_{\infty,\infty}}$ for all $k_1 < k_2$.

We have two cases depending on the condition

$$\exists\, t_{k_0} \in \{t_k\} \text{ such that } \forall\, k \geq k_0 \text{ we have } t_{k+1}' \leq t_k''. \tag{9}$$

Case 1: If (9) is true, then $[t_0', T) = \cup_{k \geq k_0} [t_k', t_k'')$. In this case let $I = [t_0', T)$. Clearly

$$\int_I \|u(t)\|_{\dot{B}^{-\epsilon}_{\infty,\infty}}^{2/(1-\epsilon)} \, \mathrm{d}t = \infty.$$

Case 2: If (9) is false then there exists an infinite sub-sequence of $\{t_k\}$, which we label $\{s_k\}$, such that $s_k'' < s_{k+1}'$ for all $k \in \mathbb{N}$. In this case let $I = \cup_{k \in \mathbb{N}} [s_k', s_k'')$. Then,

$$\int_I \|u(t)\|_{\dot{B}^{-\epsilon}_{\infty,\infty}}^{2/(1-\epsilon)} \, \mathrm{d}t \geq \sum_{k \in \mathbb{N}} \frac{T^*(s_k)}{2} \|u(s_k)\|_{\dot{B}^{-\epsilon}_{\infty,\infty}}^{2/(1-\epsilon)} = \sum_{k \in \mathbb{N}} \frac{c_0^{2/(1-\epsilon)}}{2} = \infty.$$

In either case, we have shown that (7) implies (8). □

3. Proofs of Theorems 1 and 2

Proof of Theorem 1. Fix $\epsilon \in (0, 1)$ and $T > 0$, and assume $u \in C(0, T; \dot{B}^{-\epsilon}_{\infty,\infty})$ is a Leray–Hopf weak solution to 3D NSE on $[0, T]$. Assume $t_0, \ldots, t_k$ are as in the statement of the lemma. It suffices to show

$$\|u(t_k)\|_{\dot{B}^{-\epsilon}_{\infty,\infty}} \leq \|u(t_0)\|_{\dot{B}^{-\epsilon}_{\infty,\infty}},$$

since then we re-solve at t_0 and, by local-in-time well-posedness and the weak-strong uniqueness of [8], see that u is regular at time T.

If $k = 0$, then we are done. Otherwise note that $t_1 \in (t_0', t_0'')$. Apply Lemma 4 at t_0 to conclude that

$$\|u(t_1)\|_{\dot{B}^{-\epsilon}_{\infty,\infty}} \leq \|u(t_0)\|_{\dot{B}^{-\epsilon}_{\infty,\infty}}.$$

If $k = 1$, then we are done. Otherwise, we repeat the argument and eventually obtain

$$\|u(t_k)\|_{\dot{B}^{-\epsilon}_{\infty,\infty}} \leq \|u(t_0)\|_{\dot{B}^{-\epsilon}_{\infty,\infty}},$$

which completes the proof.

Proof of Theorem 2. Assume u is a Leray–Hopf weak solution on $[0, T]$ which belongs to $C(0, T; \dot{B}^{-\epsilon}_{\infty,\infty})$.

By Lemma 3 with $M = \|u(t)\|_{\dot{B}^{-\epsilon}_{\infty,\infty}}/2$ it follows that

$$\sup_{j \leq J_{\mathrm{low}}(t)} \lambda_j^{-\epsilon} \|\dot{\Delta}_j u(t)\|_\infty < \frac{1}{2}\|u(t)\|_{\dot{B}^{-\epsilon}_{\infty,\infty}}. \tag{10}$$

If u loses regularity at time T, local well-posedness in $\dot{B}^{-\epsilon}_{\infty,\infty}$ implies that

$$\|u(t)\|_{\dot{B}^{-\epsilon}_{\infty,\infty}} \geq \left(\frac{c_*}{T - t}\right)^{(1-\epsilon)/2},$$

for a small universal constant c_*. Therefore,

$$\int_0^T \|u(t)\|^{2/(1-\epsilon)}_{\dot{B}^{-\epsilon}_{\infty,\infty}} \, \mathrm{d}t = \infty.$$

Let $\mathcal{E}$ denote the collection of escape times in $(0, T)$ and let $I = \cup_{t \in \mathcal{E}}(t', t'')$. By Lemma 6

$$\int_I \|u(t)\|^{2/(1-\epsilon)}_{\dot{B}^{-\epsilon}_{\infty,\infty}} \, \mathrm{d}t = \infty.$$

For each $t \in I$ there exists an escape time $t_0(t)$ so that $t \in (t_0', t_0'')$. Thus,

$$\frac{1}{2}\left(\frac{c_0}{\|u(t_0)\|_{\dot{B}^{-\epsilon}_{\infty,\infty}}}\right)^{2/(1-\epsilon)} \leq t - t_0 \leq \left(\frac{c_0}{\|u(t_0)\|_{\dot{B}^{-\epsilon}_{\infty,\infty}}}\right)^{2/(1-\epsilon)}.$$

By re-solving at t_0 using subcritical well-posedness, inequality (6), and weak-strong uniqueness (see [8]), we have

$$(t-t_0)^{1/2}\|u(t)\|_{\dot B^{1-\epsilon}_{\infty,\infty}} \le c_0\|u(t_0)\|_{\dot B^{-\epsilon}_{\infty,\infty}}.$$

Consequently,

$$\lambda_j^{-\epsilon}\|\dot\Delta_j u(t)\|_\infty \le 2c_0\lambda_j^{-1}\|u(t_0)\|^{1+1/(1-\epsilon)}_{\dot B^{-\epsilon}_{\infty,\infty}} \le 2c_0\lambda_j^{-1}\|u(t)\|^{1+1/(1-\epsilon)}_{\dot B^{-\epsilon}_{\infty,\infty}},$$

where we have used the fact that t_0 is an escape time. Using (1) we obtain

$$\sup_{j\ge J_{\text{high}}(t)} \lambda_j^{-\epsilon}\|\dot\Delta_j u(t)\|_\infty < \frac{\|u(t)\|_{\dot B^{-\epsilon}_{\infty,\infty}}}{2}. \tag{11}$$

Combining (10) and (11) yields

$$\int_I \left(\sup_{J_{\text{low}}(t)\le j\le J_{\text{high}}(t)} \lambda_j^{-\epsilon}\|\dot\Delta_j u(t)\|_{L^\infty}\right)^{2/(1-\epsilon)} \mathrm{d}t = \infty,$$

which proves Theorem 2.

Remark 7. If we only wanted to eliminate low frequencies in Theorem 2 then an alternative proof is available which we presently sketch. Decompose $[0,T]$ into adjacent, disjoint intervals $[t_k,t_{k+1})$ with $t_{k+1}-t_k \sim 2^{-k}T$. Then, a solution which is singular at T must satisfy

$$2^k \lesssim \|u(t\sim t_k)\|^{2/(1-\epsilon)}_{\dot B^{-\epsilon}_{\infty,\infty}}.$$

Using the Bernstein inequalities we have

$$\begin{aligned}\int_{t_k}^{t_{k+1}} \left(\sup_{j\le J_0(t)} \lambda_j^{-\epsilon}\|\dot\Delta_j u(t)\|_\infty\right)^{2/(1-\epsilon)} \mathrm{d}t &\le \int_{t_k}^{t_{k+1}} \left(\sup_{j\le J_0(t)} \lambda_j^{3/2-\epsilon}\|\dot\Delta_j u(t)\|_2\right)^{2/(1-\epsilon)} \mathrm{d}t\\ &\lesssim \|u\|^{2/(1-\epsilon)}_{L^\infty L^2}\lambda_{J_0(t)}^{(3-2\epsilon)/(1-\epsilon)}(t_{k+1}-t_k)\\ &\lesssim \|u\|^{2/(1-\epsilon)}_{L^\infty L^2} 2^{J_0(t)(3-2\epsilon)/(1-\epsilon)}2^{-k}.\end{aligned}$$

Define J_0 so that $J_0(t)(3-2\epsilon)/(1-\epsilon) = k/2$ for $t\in[t_k,t_k+1)$. Then, terms on the right hand side are summable and we obtain

$$\int_0^T \left(\sup_{j\le J_0} \lambda_j^{-\epsilon}\|\dot\Delta_j u(t)\|_\infty\right)^{2/(1-\epsilon)} \mathrm{d}t < \infty.$$

Since the integral over all modes must be infinite at a first singular time, we conclude

$$\int_0^T \left(\sup_{j\ge J_0(t)} \lambda_j^{-\epsilon}\|\dot\Delta_j u(t)\|_\infty\right)^{2/(1-\epsilon)} \mathrm{d}t = \infty.$$

Further analyzing the definition of J_0 and the lower-bound for the $\dot{B}^{-\epsilon}_{\infty,\infty}$ norm we see that

$$J_0(t) \sim \log_2 \left(\|u(t)\|^{2/(3-2\epsilon)}_{\dot{B}^{-\epsilon}_{\infty,\infty}} \right),$$

which matches the rate found using the other approach.

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Z. Bradshaw & Z. Grujić
University of Virginia
Charlottesville,
VA, USA
e-mail: zg7c@virginia.edu

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