Chapter 2

Fluid–Structure interaction model: wellposedness, regularity and control*

Viorel Barbu[1], Zoran Grujić[2], Irena Lasiecka[2], and Amjad Tuffaha[3]

[1] *Romanian Academy, Iasi, Romania*
[2] *Department of Mathematics, University of Virginia, Charlottesville, VA 22901, USA*
[3] *Department of Mathematics, University of Southern California, Los Angeles, CA, USA*

Nonlinear fluid–structure interaction coupling the Navier–Stokes equations with dynamic system of elasticity is considered. The coupling takes place on the boundary (interface) via the continuity of the normal component of the Cauchy stress tensor. Due to mismatch of parabolic and hyperbolic regularity previous results in the literature dealt with either regularized version of the model, or with very smooth initial conditions leading to local existence only. In contrast, in the case of small but rapid oscillations of the interface, this chapter reports on results ensuring (1) existence of global *finite energy* weak solutions, (2) regularity of solutions and (3) control theoretic properties of the system. This is achieved by exploiting new hyperbolic trace regularity results which provide a way to deal with the mismatch of parabolic and hyperbolic regularity.

2.1 Introduction

Interaction of a fluid/plasma and an elastic structure via an interface is a basic coupling in continuum mechanics. There are essentially two different scenarios: one in which an elastic solid is fully immersed in a fluid (e.g., a submarine submerged in an ocean or a microbubble suspended in a body fluid – used as a contrast in ultrasound imaging [17]), or adherence/detachment of leukocytes in a blood flow), and the other one is when a fluid is flowing along a pipe/filling a container with elastic walls (e.g. [18]), the flow of blood along blood vessels [10]. We focus on the first case.

* This chapter is dedicated to Prof. A. V. Balakrishnan.

However, mathematical subtleties are common to both cases. While there has been a lot of interest and attention paid to the understanding the dynamics of these structures, the vast majority of papers is devoted to numerical and experimental studies. Mathematical-PDE oriented analysis, instead, is rather scarce with many problems still unresolved. This, in particular, refers to the mathematically fundamental issues such as wellposedness of weak solutions and their regularity. The latter is the main topic of the present chapter.

In what follows we describe the model under consideration. Let $\Omega \subset R^n, n = 2, 3$, be a bounded simply connected domain with an interior region Ω_s (a domain occupied by an elastic solid) and an exterior region Ω_f (a domain filled with viscous incompressible fluid). Denote by Γ_f the outer boundary of the domain Ω_f and by Γ_s the boundary of the region Ω_s which is also an interior boundary of Ω_f, and where the interaction takes place. Let u be a vector-valued function defined on $\Omega_f \times [0, T]$ representing the velocity of the fluid and p a scalar-valued function representing the pressure. Additionally, let w, w_t be the displacement and the velocity functions of the elastic solid Ω_s. We also denote by ν the unit outward normal vector on Γ_s with respect to the region Ω_s.

We are considering the following PDE model of fluid–structure interaction defined by the variables (u, w, w_t, p).

$$\begin{cases} u_t - \operatorname{div} \mathcal{T}(u, p) + (u \cdot \nabla)u = 0 & \Omega_f \times (0, T) \\ \operatorname{div}\ u = 0 & \Omega_f \times (0, T) \\ w_{tt} - \operatorname{div}\ \sigma(w) = 0 & \Omega_s \times (0, T) \\ u(0, \cdot) = u_0 & \Omega_f \\ w(0, \cdot) = w_0,\ w_t(0, \cdot) = w_1 & \Omega_s \\ u = 0 & \Gamma_f \times (0, T) \\ w_t = u & \Gamma_s \times (0, T) \\ \sigma(w) \cdot \nu = \mathcal{T}(u, p) \cdot \nu + \frac{1}{2}(u \cdot \nu)u & \Gamma_s \times (0, T) \end{cases} \tag{2.1}$$

where Cauchy Polya fluid tensor $\mathcal{T}(u, p)$ is given by

$$\mathcal{T}(u, p) \equiv \epsilon(u) - pI$$

and the *elastic stress tensor* σ and the *strain tensor* $\epsilon(u)$, respectively, are given by

$$\sigma_{ij}(u) = \lambda \sum_{k=1}^{k=3} \epsilon_{kk}(u)\delta_{ij} + 2\mu\epsilon_{ij}(u), \quad \lambda, \mu > 0, \quad \text{and} \quad \epsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Note that the continuity of both the velocities and the normal components of the stress tensors across the interface is required.

In the system considered, the interface Γ_s is stationary. This corresponds to a physical situation in which the order of magnitude of the displacement of the elastic solid on the boundary is smaller than the order of magnitude of the velocity (small but rapid oscillations) [14, 29]. If one considers the case of a moving interface, the equation for the interface in the Lagrangian coordinates is comparable to our elastic equation [12], and the core of the problem arising from the parabolic–hyperbolic coupling across the interface via the continuity of the velocities and the normal components of the stress tensors is similar.

Also, note that the presence of the – not necessarily small – fluid term $\frac{1}{2}(u \cdot \nu)u$ on Γ_s is due to the fact that the interface is stationary. Namely, in addition to the normal component of the fluid Cauchy stress tensor $\mathcal{T} = -pI + \epsilon(u)$ where ϵ is the deformation tensor, which would be present in the case of the moving interface as well, this model features an additional stress exerted on the interface originating in the tendency of the fluid to advect through the interface. The advection term in the Navier–Stokes equations is $(u \cdot \nabla)u$ which is, due to the incompressibility of the fluid, equal to div $(u \otimes u)$, and that corresponds exactly to a boundary term of the $(u \otimes u)\nu = (u \cdot \nu)u$-type (see also [29]). In the case of a moving interface [12], this boundary term is entirely absorbed by the material derivative of the velocity of the fluid.

This model (with both moving and stationary interface) has been well-established in both mathematical and modeling literature – see, e.g., [12, 14, 15, 16, 27, 31] – and the applications range from naval and aerospace engineering to cell biology and biomedical engineering [10, 18]. However, due to the nature of this particular form of parabolic-hyperbolic coupling, even the basic question of existence of the natural energy-class weak solutions had not been previously resolved. A key issue is that the traces of the elastic (wave/hyperbolic) component at the energy level are not defined via the standard trace theory.

There had been two different avenues – effective also in the case of moving interface – in approaching this problem. The first one had been to add a "structural damping" term, thus effectively *regularizing* the elastic/hyperbolic dynamics (cf. [9, **?**] and the references therein) and clearing the stage for the standard trace theory to apply. In this approach, the heart of the difficulty related to boundary traces is essentially defined away. The other avenue had been to consider the case of *very smooth* data–this led to a functional setting in which the standard trace theory applies yielding local-in-time existence of smooth solutions (cf. [12, 15]). However, when dealing with the original nonregularized model within the framework of finite energy solutions, with moving or stationary interface, the main mathematical obstacles remain to be:

1. Mismatch between parabolic and hyperbolic regularity which is most pronounced on the interface where the traces of hyperbolic solutions are not *a priori* defined in the topology of finite energy space.
2. The presence of Neumann type boundary conditions that rules out standard approaches to the NSE equations via Leray's projection.
3. The coupling between fluid and the structure taking place on the interface- boundary, hence contributing to the issue of mismatch of regularity between the two types of dynamics.

The main aim of this chapter is to report on recent results and methods developed for the analysis and control of fluid–structure interactions. In particular, we shall present several results pertaining to (i) existence of *weak* -finite energy solutions, (ii) their regularity and (iii) control problems associated with finite energy s olutions.

In a very recent work [4], the issues raised above in the case of stationary interface have been successfully dealt with and the authors established global-in-time existence of the energy-level weak solutions to (2.1) *without any* regularization of the elastic/hyperbolic dynamics. One of the key ingredients was establishment of an *improved* (often referred as "hidden" [28]) trace regularity of hyperbolic solutions, that provides a way to deal with the mismatch of the regularity. As a consequence, the functional spaces for the fluid component are exactly the same as in the classical Leray theory of weak solutions for the NSE *per se*.

Throughout the chapter $\mathcal{H} \equiv H \times H^1(\Omega_s) \times L_2(\Omega_s)$ where

$$H \equiv \{u \in L_2(\Omega_f)\colon \text{div } u = 0, u \cdot \nu|_{\Gamma_f} = 0\}$$

will denote the energy space for the system.

Note that all Sobolev spaces H^s, L_2 pertaining to u and w are in fact $(H^s)^n$, $(L_2)^n$, $n = 2, 3$ and only for simplicity we omit the exponent n.

In addition we will use the following notation:

$$V \equiv \{v \in H^1(\Omega_f)\colon \operatorname{div} v = 0, u|_{\Gamma_f} = 0\}.$$

By $C_w([0,T], X)$ we denote a space of functions that are weakly continuous in t with values in X.

$$(u, v)_s \equiv \int_{\Omega_s} uv dx, \quad \langle u, v\rangle \equiv \int_{\Gamma_s} uv d\sigma.$$

C will denote a generic constant, which can be different at different occurrences.

Finally, the energy functional for the system is given by

$$E(t) \equiv |u(t)|^2_{0,\Omega_f} + (\sigma(w(t)), \epsilon(w(t)))_s + |w_t(t)|^2_{0,\Omega_s}.$$

2.2 Weak solutions

We begin by defining weak solutions to the original system (2.1). This is obtained by projecting the equations to H and utilizing the boundary conditions.

Definition 2.1. Weak solution *Let $(u_0, w_0, w_1) \in \mathcal{H}$ and $T > 0$. We say that a triple $(u, w, w_t) \in C_w\left([0,T]; H \times H^1(\Omega_s) \times L_2(\Omega_s)\right) = C_w([0,T]; \mathcal{H})$, $u \in L_2((0,T); V)$ is a weak solution of (2.1) if*

- $(u(0), w(0), w_t(0)) = (u_0, w_0, w_1)$ *(in the sense of weak continuity),*
- $\sigma(w).\nu \in L_2((0,T); H^{-1/2}(\Gamma_s))$,
- $\frac{d}{dt} w|_{\Gamma_s} = u|_{\Gamma_s} \in \ L_2((0,T), H^{1/2}(\Gamma_s))$
- *and the following variational system holds a.e. in $t \in (0,T)$:*

$$\begin{aligned} &\frac{d}{dt}(u_t, \phi)_f + (\epsilon(u), \epsilon(\phi))_f + ((u \cdot \nabla)u, \phi)_f \\ &\qquad + \langle \sigma(w) \cdot \nu, \phi\rangle - \frac{1}{2}\langle (u.\nu)u, \phi\rangle = 0 \\ &\frac{d}{dt}(w_t, \psi)_s + (\sigma(w), \epsilon(\psi))_s - \langle \sigma(w).\nu, \psi\rangle = 0 \end{aligned} \tag{2.2}$$

for all test functions $\phi \in V$ and $\psi \in H^1(\Omega_s)$.

Remark 2.1. 1. Note that weak solutions, as defined in Definition 2.1, require information on the trace $\sigma(w).\nu|_{\Gamma_s}$. This does not follow from the interior regularity of finite energy solutions. Thus, the definition of *weak* solutions imposes *an additional regularity* requirement on the normal stresses of solid's displacement w on the interface. The fact that such requirement is necessary, follows from the variational principle used with independent test functions ϕ, ψ which are not required to match on the interface [31], hence they retain normal stresses in the formulation. In short, the presence of $\sigma(w) \cdot \nu$ in the boundary terms is an intrinsic feature of the definition of weak solutions which, in turn, requires imposition of the trace regularity postulated by Definition 2.1. The key point we want to make is that this additional regularity (which does not follow from any trace theory) is shown to be a *property* of finite energy solutions, rather than an artifact assumed arbitrarily on solution. This will be documented below.

2. The time derivative of the trace of w, appearing in the condition expressing matching of velocities, is understood in the sense of distribution. The intrinsic regularity of the fluid on the boundary allows to identify the distribution with $L_2(0, T, H^{1/2}(\Gamma_s))$ function.

A starting point of our analysis is existence result for weak and global solutions obtained in [4].

Theorem 2.1. Existence of weak solutions. *Given any initial condition* $(u_0, w_0, w_1) \in \mathcal{H}$ *and any* $T > 0$*, there exists a weak solution* (u, w, w_t) *to the system (2.1) such that*

$$\nabla w|_{\Gamma_s} \in L_2((0,T); H^{-1/2}(\Gamma_s))$$
$$\frac{d}{dt} w|_{\Gamma_s} = w_t|_{\Gamma_s} \in L_2((0,T); H^{1/2}(\Gamma_s)) \tag{2.3}$$

Moreover, in the case when dimension of $\Omega = 2$*, weak solutions are unique within the class specified above.*

As mentioned before, a weak solution as defined in Definition 2.1 requires information on the trace $\sigma(w).\nu|_{\Gamma_s}$, which does not follow from finite energy regularity of solutions. Fortunately, Theorem 2.1 does provide existence of finite energy solutions with the *additional* boundary regularity. This confirms that the definition of *weak* solution is a correct one for the problem under consideration.

The key result used in order to establish additional boundary regularity in (2.3) is the following trace regularity of finite energy solutions to a linear elastic wave equation.

Lemma 2.1. [[4]] *Let* (w, w_t) *be a solution to an elastic wave equation defined distributionally on* $\Omega \times (0, T)$

$$w_{tt} - div\, \sigma(w) = 0$$

driven by the folllowing data:

$$w(0) \in H^1(\Omega_s), \quad w_t(0) \in L_2(\Omega_s), \quad \frac{d}{dt} w|_{\Gamma_s} \in L_2(0,T; H^{1/2}(\Gamma_s)).$$

Then $\frac{\partial}{\partial \nu} w \in L_2((0,T) \times \Gamma_s) \oplus C([0,T); H^{-1/2}(\Gamma_s))$.

Remark 2.2. We note that the trace result stated in Lemma 2.1 does not follow from interior regularity of solutions to the wave equation. It is an independent regularity result, inspired by techniques developed in [7, 23, 28], and obtained by microlocalizing the problem to"hyperbolic" and "elliptic" sectors which represent, respectively , $L_2((0,T) \times \Gamma_s)$ and $C([0,T); H^{-1/2}(\Gamma_s))$ regularity of the normal derivatives.

The goal of this chapter is to show that if the initial data are sufficiently smooth and satisfy natural compatibility conditions on the interface Γ_s, the weak solutions obtained in Theorem 2.1 are in fact smooth. The main results will be stated in the following section, and the rest of the chapter is devoted to the proofs.

The proof of Theorem 2.1 given in [4] follows through several major steps which are outlined below:

Step 1. We first consider Lipschitz truncation of nonlinear term, referring to the resulting problem as P_n. It is shown that the projection of P_n onto the space V leads to a Lipschitz perturbation of a maximal monotone problem. The theory of maximal monotone operators is used [2] in order to assert existence and uniqueness of the corresponding solutions. The obtained solution is nonlinear semigroup solution.

Step 2. In the next step we prove that the semigroup solution to truncated problem is also variational solution. The key element of this step is a correct identification of the normal stresses on the boundary. The latter are not defined via the topology of the underlying energy spaces. This brings us to the next step.

Step 3. Here the main challenge is to identify normal stresses as $H^{-1/2}$ elements on the boundary. This is achieved by using "sharp" trace regularity given in Lemma 2.1.

Step 4. Having obtained variational form of the truncated problem, the last step is passage with the limit on the variational formulation of the truncated problem. Here agian, weak compactness methods along with sharp regularity of the trace are critical.

2.3 Strong solutions

Strong solutions refer to the original PDE system and they are defined as follows.

Definition 2.2. Strong solutions. *We say that (u, w, w_t, p) is a strong solution of (2.1) if*

- $(u, w, w_t) \in L_\infty((0,T); V \times H^2(\Omega_s) \times H^1(\Omega_s))$
- $(u, p) \in L_2((0,T); H^2(\Omega_f) \times H^1(\Omega_f))$
- $(u_t, w_t, w_{tt}) \in L_\infty((0,T); H \times H^1(\Omega_s) \times L_2(\Omega_s)) = L_\infty((0,T); \mathcal{H})$
- *Strong form of equations given in (2.1) holds a.e in $\Omega \times (0,T)$.*

As expected, in order to be able to obtain strong solutions, one must impose a suitable compatibility conditions on the initial data. These are formulated below.

Definition 2.3. Compatibility Conditions (CC). *We say that initial conditions $(u_0, w_0, w_1) \in H^2(\Omega_f) \cap V \times H^2(\Omega_s) \times H^1(\Omega_s)$ satisfy Compatibility Conditions (CC) if*

- $w_1 = u_0,$ *on* Γ_s
- $\langle \sigma(w_0).\nu - \epsilon(u_0).\nu + 1/2(u.\nu)u, \phi \rangle = 0$ *for all* $\phi \in V$.

In order to formulate our results, we shall distinguish two- and three-dimensional domains.

2.3.0.1 The two-dimensional case

Theorem 2.2. *Let $\Omega \subset R^2$. Then given $(u_0, w_0, w_1) \in H^2(\Omega_f) \bigcap V \times H^2(\Omega_s) \times H^1(\Omega_s)$ such that the compatibility conditions (CC) are satisfied, the following holds. A weak solution asserted* *by Theorem 2.1 becomes a strong solution* (u, w, w_t, p) *satisfying the system (2.1). Moreover,* $u \in C([0, T]; V)$ *and the strong solutions are unique.*

Remark 2.3. We note that the H^2 regularity of fluid component is only L_2 in time, rather than L_∞ – as in the classical Navier–Stokes equations. This is due to topological mismatch between the fluid and the solid – a feature that characterizes the interaction.

2.3.0.2 The Three-Dimensional Case

In the three-dimensional case we shall consider two different situations. Local-in-time strong solutions for arbitrary large initial data and global-in-time strong solutions for small initial data.

1. Local-in-time strong solutions for general initial data

Theorem 2.3. *Let* $\Omega \subset R^3$*. Then given* $(u_0, w_0, w_1) \in H^2(\Omega_f) \bigcap V \times H^2(\Omega_s) \times H^1(\Omega_s)$ *such that the compatibility conditions (CC) are satisfied, there exists* $T' > 0$ *such that a weak solution on* $(0, T')$ *becomes a strong solution* (u, w, w_t, p) *satisfying the system (2.1). Moreover,* $u \in C([0, T']; V)$ *and the strong solutions are unique.*

2. Global-in-time strong solutions for small initial data

Theorem 2.4. *Let* $\Omega \subset R^3$*, and* $(u_0, w_0, w_1) \in H^2(\Omega_f) \bigcap V \times H^2(\Omega_s) \times H^1(\Omega_s)$ *such that the compatibility conditions (CC) are satisfied. Assume that*

$$|u_0|^2_{2,\Omega_f} + |w_0|^2_{2,\Omega_s} + |w_1|^2_{1,\Omega_s} \leq C$$

for a suitable absolute constant C*. Then, there exists a unique strong solution* (u, w, w_t, p)*,* $u \in C([0, \infty); V)$*, satisfying the system (2.1) on* $(0, \infty)$*.*

Strong solutions correspond to the original PDE – hence they involve the pressure term p. Having proved existence of weak solutions in [4], the next step is to analyze regularity of these solutions given smooth initial data satisfying the compatibility conditions (CC). The proof of regularity/smoothness given in [6] relies on the following major steps:

Step 1. We prove that *time derivatives* are also bounded in the finite energy space $\mathcal{H}$.

Step 2. Time regularity of weak solutions allows for a reconstruction of the PDE form and, in particular, for the identification of the pressure.

Step 3. In this step, we aim at obtaining higher space regularity. This step consists of two substeps. First we prove the additional regularity of the tangential derivatives of solutions defined in a collar neighborhood of the interface Γ_s. In the second step we reconstruct full H^2 regularity of solutions by appealing to a version of Agmon–Douglis–Nirenberg methods. Thus, at the end of the process we obtain regular (classical) solutions corresponding to the original problem equipped with“smooth” and compatible initial conditions.

Remark 2.4. The results presented above pertain to *static* interface model. However, it is our belief that the techniques developed here should have strong bearing on the theory of weak and strong wellposedness for the corresponding *moving* interface problem which, to the best of our knowledge, is an open and very challenging issue.

2.4 Control problem

Once the wellposedness of both weak and strong solutions is established, a natural issue to consider are control problems associated with the dynamics in (2.1). This includes both stabilization and optimal control problems. Of particular interest are boundary control problems where control function located on an interface between the two media is expected to alter dynamic behavior of the structure. For instance stabilization of the structure in the neighborhood of unstable equilibria, or a minimization of the overall energy while reaching a preassigned target for the velocity of the fluid are common examples of classical control problems that are of interest in fluid dynamics and, more generally, engineering. Stabilization problems in the vicinity of unstable equilibria were considered for the two- and three-dimensional Navier Stokes in [5] and references therein. This chapter, instead, is focused on optimal control problems governed by a linearization (around unstable neighborhood) of nonlinear dynamics.

This leads to consideration of the following linearized dynamics:

$$\text{fluid} \quad \begin{cases} u_t - \operatorname{div} \mathcal{T}(u,p) + L_1(t)\nabla u + L_2(t)u = 0, & \Omega_f \times (0,T) \\ \operatorname{div} \ u = 0, & \Omega_f \times (0,T) \end{cases} \tag{2.4}$$

$$\text{solid} \quad \begin{cases} w_{tt} - \operatorname{div} \ \sigma(w) = 0, & \Omega_s \times (0,T) \end{cases} \tag{2.5}$$

$$\text{initial data} \quad \begin{cases} u(0,\cdot) = u_0 & \Omega_f \\ w(0,\cdot) = w_0, \ w_t(0,\cdot) = w_1 & \Omega_s \end{cases} \tag{2.6}$$

$$\text{boundary data} \quad \begin{cases} u = 0 & \Gamma_f \times (0,T) \\ w_t = u & \Gamma_s \times (0,T) \\ \sigma(w)\cdot\nu = \mathcal{T}(u,p)\cdot\nu + g & \Gamma_s \times (0,T) \end{cases} \tag{2.7}$$

where $L_1(t), L_2(t)$ are time dependent smooth coefficients resulting from the linearization of the advection term in the equation. The function $g \in L_2([0,T] \times \Gamma_f)$ is a control *force* acting on the interface.

2.4.1 Semigroup framework

We write (2.4) in an abstract form as

$$Y_t = A(t)Y + Bg, \ \ \textit{for} \ \ Y(0) = Y_0 \in \mathcal{H} \tag{2.8}$$

where the state $Y(t) \equiv (u(t), w(t), w_t(t))$. The control force $g \in L_2((0,T) \times \Gamma_s)$.

The operator $A(t) = A_0 + L(t)$ is given by

$$A_0(Y) \equiv \begin{pmatrix} \mathcal{A}(u, \sigma(w)) \\ z \\ div(\sigma(w) \end{pmatrix}$$

with $\mathcal{A} : V \times H^{-1/2}(\Gamma) \to V'$ defined as

$$(\mathcal{A}(u, W_t), \phi)_{V,V'} = (\epsilon(u), \epsilon(\phi))_f + \langle z \cdot \nu, \phi \rangle, \quad \text{for all } \phi \in V$$

$$D(A_0) \equiv \{(u, w, z) \in \mathcal{H}: \mathcal{A}(u, \sigma(w)) \in H, z \in H^1(\Omega_s)$$

$$\operatorname{div} \sigma(w) \in L_2(\Omega_s), z|_{\Gamma_s} = u|_{\Gamma_s}\}$$

$$L(t)Y = \begin{pmatrix} L_1(t)\nabla u + L_2(t)u \\ 0 \\ 0 \end{pmatrix}$$

The control operator has the structure

$$Bg \equiv [\mathcal{B}g, 0, 0]$$

with boundary operator γ defined through its adjoint $(\mathcal{B}g, \phi)_f = \langle g, \phi \rangle, \forall \phi \in V$

It was shown in [4] that A_0 generates C_0 semigroup of contractions on $\mathcal{H}$. The same argument as used in [4] shows that $A(t)$ generates strongly continuous evolution on $\mathcal{H}$.

2.4.2 Control problem

With the dynamics in (2.8) we associate the functional cost given by

$$J(Y, g) \equiv \int_0^T |RY(t)|^2_{\mathcal{H}} + |g(t)|^2_{L_2(\Gamma_s)} dt + |G(Y(T) - Y_T)|^2_{\mathcal{H}},$$

where R and G are appropriate weights (bounded operators on $\mathcal{H}$) and Y_T is a preassigned target.

If the control operator B were bounded on $\mathcal{H}$, then optimal control problem associated with (2.8) would be just a standard optimal control problem. The issue, however, is that the control operator B is not bounded. While *existence* and uniqueness of the optimal pair Y^0, g^0 still can be shown by using tools of convex analysis,optimal feedback synthesis and the associated Riccati (or H-Jacobi) equations are much more subtle problems. It is known (with counterexamples, [30, 32]) that unbounded control operators may affect, in general, wellposedness of the Riccati feedback synthesis and solvability of Riccati equations. The exception to this are generators of analytic semigroups, which possess an inherent smoothing effect. For this class of problems a complete Riccati theory has been already established [3, 8, 24]. However, fluid structure inetraction problem under consideration , which consists of parabolic-hyperbolic coupling – does *not* enjoy analyticity property. This raises a very fundamental issue: do we have optimal feedback synthesis for the control problem governed by dynamics in (2.8)? In other words, do we have a meaningful feedback representation of the form

$$g^0(t) = -B^* P(t) Y^0(t), \quad `t > 0 \tag{2.9}$$

where the operator $P(t) \in \mathcal{L}(\mathcal{H}))$, (value function) satisfies a suitable nonstandard Riccati equation. Since B^* is an unbounded operator, the composition $B^* P(t)$ may not be even densely defined (as shown by examples [30, 32]). Thus, the very definition of the composition operator $B^* P(t)$ lies in the heart of the problem.

2.4.3 Singular estimate

The main purpose of this section is to assert a positive answer to the question raised above. A central role in this issue is played by the so-called "singular estimate" satisfied by the pair $[A_0, B]$. This is to say: there exists $0 \leq \gamma < 1$ such that

$$|e^{A_0 t} B g|_{\mathcal{H}} \leq \frac{C}{t^{\gamma}} |g|_{L_2(\Gamma_s)}, \quad 0 < t \leq 1 \tag{2.10}$$

The bound of the type as in (2.10) is typical for analytic semigroups [8, 24]. In fact, singular estimate was first introduced by Balakrishnan [3], in the context of Dirichlet control of heat equation, and has been referred since then as Balakrishnan–Washburn bound. This was a critical development in the treatment of boundary control models [3]. It turns out that this bound is critical in establishing many control theoretic properties of systems with unbounded control actions. Wellposedness of feedback synthesis, existence of the associated Riccati equations and regularity of the transfer function are prime examples. The analysis of control systems which satisfy Singular Estimate bound is given in [8, 24] – in the case of analytic semigroups and in [19, 22] in the nonanalytic case.

The main result reported in this section is that fluid structure interaction problem does satisfy singular estimate property [20].

Theorem 2.5. *With reference to control system (2.8). singular estimate is satisfied with the value* $\gamma = 1/4 + \epsilon$, *for every* $\epsilon > 0$.

Remark 2.5. The singular estimate established by Theorem 2.5 leads to well defined *control-to-state map*

$$(Lg)(t) \equiv \int_0^t e^{A(t-s)} B g(s) ds$$

which is bounded $L_2((0, T) \times \Gamma_f) \to \mathcal{H}$. This represents a "hiidden" regularity property of the control system that does not follow from the semigroup approach.

Singular estimate in Theorem 2.5 opens a door for the rigorous analysis of control problem and allows to establish wellposed feedback synthesis (2.9) along with the wellposedness of Riccati equations associated with the control problem under consideration. In particular, one obtains the following singular estimate for the gain operator $B^* P(t)$.

$$|B^* P(t)|_{\mathcal{H} \to L_2(\Gamma_s)} \leq \frac{C}{(T-t)^{1/4+\epsilon}}$$

The details are given in [20] see also [21].

References

[1] S. Agmon, A. Douglis and L. Nirenberg. Estimates near the boundary for solutions of elliptic partial differential quations satisfying general boundary value conditions, I, *Comm. Pure Appl. Math.* **12** (1959), 623–727.
[2] V. Barbu. *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Nordhoff, 1975.
[3] A. Balakrishnan. *Applied Functional Analysis*, Springer Verlag, 1975.

[4] V. Barbu, Z. Grujić, I. Lasiecka and A. Tuffaha. Existence of the Energy-Level Weak Solutions for a Nonlinear Fluid–Structure Interaction Model, Contemporary Mathematics 440, Fluid and Wave Interactions, 55–82 (2007).

[5] V. Barbu, I. Lasiecka and R. Triggiani. *Tangential Boundary Stabilization of Navier Stokes Equations* , Memoires AMS, Nr 852, 2006.

[6] V. Barbu, Z. Grujic, I. Lasiecka and A. Tuffaha. Smoothness of weak solutions to nonlinear fluid-structure intercation model, *Indiana University Mathematics Journal*, **57**(3) (2008), 1173–1207.

[7] M. I. Belishev and I. Lasiecka. *The Dynamical Lame System: Regularity of Solutions, Boundary Controllability and Boundary Data Continuation, ESAIM: Control, Optimisation and Calculus of Variation*, **8** (2002), 143–167.

[8] A. Bensoussan, G. Da Prato, M. C. Delfour and S. K. Mitter. *Representation and Control of Infinite Dimensional Systems*, Birkhauser.

[9] H. Beirao Da Veiga. On the Existence of Strong Solutions to a Coupled Fluid-Structure Evolution Problem, *Journal of Mathematical Fluid Mechanics* **6** (2004), 21–52.

[10] Mikelic, G. Giodoboni, S. Canic. Fluid–Structure Interaction in a Pre-Stressed Tube with Thick Elastic Walls I: The Stationary Stokes Problem. *Networks and Heterogeneous Media* **2**(3) (2007), 397–423.

[11] P. Constantin and C. Foias. *Navier–Stokes Equations*. The University of Chicago Press, Chicago 1988.

[12] D. Coutand and S. Shkoller. Motion of an elastic solid inside an incompressible viscous fluid. *Arch. Ration. Mech. Anal.* **176**(1) (2005), 25–102.

[13] B. Desjardins, M. J. Esteban, C. Grandmont and P. Le Tallec. Weak solutions for a fluid-elastic structure interaction model. *Rev. Mat. Complut*. **14**(2) (2001), 523–538.

[14] Q. Du, M. D. Gunzburger, L. S. Hou and J. Lee. Analysis of a linear fluid–structure interaction problem. Discrete Continuous Dynam. Syst. **9**(3) (2003), 633–650.

[15] E. Feireisl. On the motion of rigid bodies in a viscous incompressible fluid. *J. Evol. Equ.* **3** (2003), 419–441.

[16] M. A. Fernandez and M. Moubachir. An exact Block–Newton algorithm for solving fluid-structure interaction problems. C.R. Acad. Sci Paris, Ser. I **336** (2003), 681–686.

[17] C. Dayton, J. Allen and K. Ferrara. The magnitude of radiation force on ultrasound contrast agent. *J. Acoust. Soc. Am.* **112**(5) (2002), 2183–2192.

[18] R. Glowinski, T. Pan, T. Hesha, D. Joseph and J. Periaux. A fictious domain approach to the direct numerical simulation of incompressible viscuous flow past moving rigid bodies: applications to particulate flow. *J. Comput. Phys.* **169** (2001), 363–426.

[19] I. Lasiecka and A. Tuffaha. Riccati equations in the Bolza problem arising in boundary control problems governed by semigroups satisfying singular estimate. *Journal of Optimization Theory and Applications (JOTA)*, **136** (2008), 229–246.

[20] I. Lasiecka and A. Tuffaha. Riccati equations arising in boundary control of fluid structure interactions. *Comput. Sci. Math.* **1** (2007), 128–147.

[21] I. Lasiecka and A. Tuffaha, Riccati theory and singular estimates for a Bolza control problem arising in linearized fluid structure interaction. *Systems and Control Letters*, **58** (2009), 409–509.

[22] I. Lasiecka and R. Triggiani. Optimal control and differential riccati equations under singular estimates for $e^{At}B$ in the absence of analyticity. *Advances in Dynamics and Control*, Special Volume dedicated to A. V. Balakrishnan, Chapman & Hall/CRC Press, pp. 271–309, 2004.

[23] I. Lasiecka and R. Triggiani. Sharp regularity results for second order hyperbolic equations of Neumann type. *Annali di Matemetica Pura et Applicata*, (IV) **CLVII** (1990), 285–367.

[24] I. Lasiecka and R. Triggiani. *Control Theory for Partial Differential Equations: Continuous and Approximations Theories*, Vol !, Volume II; Cambridge.

[25] I. Lasiecka and R. Triggiani. *Uniform stabilization of the wave equation with dirichlet or neumann feedback control without geometrical conditions*. *Appl. Math. Optim.* **25** (1992), 189–224.

[26] I. Lasiecka and R. Triggiani. *Sharp regularity theory for elastic and thermoelastic kirchoff equations with free boundary conditions*. Rocky Mountain Journal of Mathematics, **30**(3) (2000).

[27] J. L. Lions and E. Magenes. *Non-Homogeneous Boundary Value Problems and Applications*. Springer-Verlag, 1972.

[28] I. Lasiecka, J. L. Lions and R. Triggiani. Non-homogeneous boundary value problems for second order hyperbolic operators. J. Mathe. Pure Appli. **65** (1986), 149–192.

[29] J. L. Lions. *Quelques methods de resolution des problemes aux limits nonlinearies*. Dunod. Paris, 1969.

[30] R. Triggiani. The Algebraic Riccati Equations with Unbounded Coefficients, Hyperbolic Case Revisited', *Contemporary Mathematics: Optimization Methods in PDE's*. 209, AMS, Providence, RI, pp. 315–339, 1997.

[31] M. Moubachir and J. Zolesio. *Moving Shape Analysis and Control: a pplications to Fluid Structure Interactions*, Chapman & Hall/CRC, 2006.

[32] G. Weiss and H. Zwart. An example of optimal control. *Syst. Control Lett.* **8** (1998), 339–349.

[33] R. Temam. *Navier–Stokes Equations*. Studies in Mathematics and its Applications, North-Holland, 1977.